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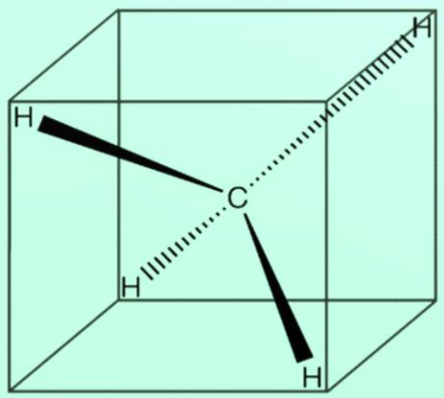
MATHEMATICS



CBSE

GRADE 12

A BOOK OF OBJECTIVE TYPE QUESTIONS

K MANI

MATHEMATICS

CBSE GRADE 12

2021

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A BOOK OF OBJECTIVE TYPE QUESTIONS

PART OF KNOW YOUR MATHS PROJECT

Dedicated

to

The Great souls who devoted their life to the Creation of the subject of Mathematics

Preface

/ Note to the Student /

This book is designed to assess the concept clarity, problem solving ability and the requisite visualisation of some key elements of Mathematics.
It is expected that the student tries the contents of this book after making the texts thorough. It is claimed that a majority of the problems can be within the reach of any reasonably prepared student.

Remember that the path from ignorance to knowledge in any subject is not straight and true, but is almost always rather zigzagged. So, upon first reading , if you are not sure of solving any problem then the best way to approach the problem is by engaging in discussion.

The book consists of 700 very short answer type questions and 40 case study questions. The case study questions consist of real life applications which justify mathematics influences every other subject. Also, these questions are a stimulant for the students to know more of Mathematics.

The number of questions had to be limited considering the size of the book. Also, I had to resist my temptation of adding more diagrams as the diagrams consume much space. Still, it is hoped that, thinking of the extensiveness of the type of questions, a student can be assured if he or she feels confident that the problems are within reach.

As no amount of books can replace a teacher, discuss as much as possible to be successful in the subject.

Add to the notion that Mathematics should be learnt only by doing, visualisation of the ideas of the subject gains priority at this level. That is, if you aim to be good at it, then DREAM MATHEMATICS.

In case you encounter any errors, please do convey to me.

With best wishes ,

K. Mani

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Salem, TamilNadu.

CONTENTS

I. RELATIONS AND FUNCTIONS

VERY SHORT ANSWER TYPE

1. Define an empty relation on N.
2. Define an empty relation on R.
3. Define a universal relation on N.
4. Define a universal relation on R.
5. Define the identity relation on A = { 1,2,3}.
6. Is the identity relation on {1,2,3} reflexive. If not, give the reason.
7. Is the identity relation on {1,2,3} symmetric. If not, give the reason.
8. Is the identity relation on {1,2,3} transitive. If not, give the reason.
9. Let A = {1,2,3}. Define a relation R by R = { (a,b) : a, b are co-primes}

   Check whether R is reflexive.
10. Let A = {1,2,3}. Define a relation R by R = { (a,b) : a, b are co-primes}

    Check whether R is symmetric.
11. Let A = {1,2,3}. Define a relation R by R = { (a,b) : a, b are co-primes}

    Check whether R is transitive.
12. Let A = {1,2} and a relation on R given by R = {(1,1),(2,2),(1,2)}. Make R, an identity relation by adding or removing ordered pair.
13. Let A = {1,2,3} and a relation on R given by

    R = {(a,b): difference between a and b is 1}. Is R reflexive ? symmetric ? transitive ?
14. Let A = {1,2,3,4,5,6,7}. Define a relation R on A by

    R = {(a,b) : both a and b are either odd or even }.

    Write the equivalence class of [1].

15. Let A = {1,2,3,4,5,6,7}. Define a relation R on A by

R = {(a,b) : both a and b are either odd or even }.

Write the equivalence class of [2].

16. Let A = {1,2,3,4,5,6,7}. Define a relation R by R = { (a,b) : b > 2a }

Check whether R is reflexive.

17. Let A = {1,2,3,4,5,6,7}. Define a relation R by R = { (a,b) : b > 2a }

Check whether R is symmetric.

18. Let A = {1,2,3,4,5,6,7}. Define a relation R by R = { (a,b) : b > 2a }

Check whether R is transitive.

19. Let A = {1,2,3,4,5,6,7}. Define a relation R on A by

R = {(a,b) : a is a divisor of b}. Is R reflexive ?. If not, give the reason.

20. Let A = {1,2,3,4,5,6,7}. Define a relation R on A by

R = {(a,b) : a is a divisor of b}. Is R symmetric ?. If not, give the reason.

21. Let A = {1,2,3,4,5,6,7}. Define a relation R on A by

R = {(a,b) : a is a divisor of b}. Is R transitive ?. If not, give the reason.

22. How many reflexive relations can be defined on a set A if n(A) = 4 ?.

23. How many symmetric relations can be defined on a set A if n(A) = 4 ?.

24. Find whether the function f : $R \rightarrow R$ defined by f(x) = e^x is onto.

25. Find whether the function f : $R - \{0\} \rightarrow R$ defined by

$f(x) = \frac{1}{x^2}$ *is one − one.*

26. Find whether the function f : $R - \{0\} \rightarrow R$ defined by

$f(x) = \frac{1}{x^2}$ *is onto.*

27. Assume that the linear function f(x) = 2x + 3 is defined with domain

" the set of positive real numbers ". Find the smallest possible

co-domain in which f(x) is surjective.

28. A linear function $y = f(x) = ax + b$ is defined where a,b are two given constants. Mention a suitable domain and co-domain where the function is onto.
29. A function $f : N \times N \to N$ is defined by $f(x,y) = xy$. Find whether f is one-one.
30. A function $f : N \times N \to N$ is defined by $f(x,y) = xy$. Find whether f is onto.
31. Is the function $f : R \to R$ defined by $f(x) = x^{1/3}$ is onto ?
32. Is the function $f : R \to R$ defined by $f(x) = x^{1/3}$ is one - one ?
33. Find the number of onto functions from a set A to a set B where $n(A) = 3, n(B) = 2$.
34. If A and B are two sets with $n(A) = 3$, $n(B) = 3$, find the number of one – one functions which can be defined from A to B.
35. Find whether the function $f : R \to R$ defined by $f(x) = x^2$ is onto.
36. Find whether the function $f : R \to R$ defined by $f(x) = x^2$ is one - one.
37. Find whether the function $f : (-\infty, 0] \to [0, \infty)$ defined by $f(x) = x^2$ is onto.
38. Find whether the function $f : (-\infty, 0] \to [0, \infty)$ defined by $f(x) = x^2$ is one – one.
39. Find whether the function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^2$ is onto.
40. Find whether the function $f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^2$ is one – one.
41. Find whether the function $f : R \to R$ where $f(x) = 4x^3 + 5$ is onto.
42. Find whether the function $f : R \to R$ where $f(x) = 4x^3 + 5$ is one-one.
43. Find whether the function $f : R - \{2\} \to R$ where $f(x) = \frac{x^2}{x+2}$ is one-one.

44. Find whether the function f:$R-\{1\} \rightarrow R$ where $f(x) = \frac{x}{x+1}$ is one-one.

45. Find whether the function f : $R-\{1\} \rightarrow R$ $f(x) = \frac{x}{x+1}$ is onto.

46. Find the range of the function f : $R-\{1\} \rightarrow R$ defined by $f(x) = \frac{x}{x+1}$.

47. Mention a suitable domain and co-domain in which the function f(x) = sin x is one – one and onto.

48. Mention a suitable domain and co-domain in which the function f(x) = sin x is one – one but not onto.

49. Mention a suitable domain and co-domain in which the function f(x) = sin x onto but not one – one.

50. Mention a suitable domain and co-domain in which the function f(x) = sin x is neither one – one nor onto.

51. Give an example of a function which is both one – one and onto.

52. Give an example of a function which is neither one – one nor onto.

I. RELATIONS AND FUNCTIONS - SOLUTIONS

1. DO IT YOURSELF.
2. DO IT YOURSELF.
3. DO IT YOURSELF.
4. DO IT YOURSELF.
5. The identity relation is R = { (1,1),(2,2),(3,3) }.
6. The identity relation is reflexive.[In fact, it is equivalence].
7. The identity relation is symmetric.[In fact, it is equivalence].
8. The identity relation is transitive.[In fact, it is equivalence].
9. A = {1,2,3}. R = { (a,b) : a, b are co-primes}

   = { (1,1),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2) }.(2,2) $\notin R$. R is not reflexive.
10. A = {1,2,3}. R = { (a,b) : a, b are co-primes}

    = { (1,1),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2) }. R is symmetric.
11. A = {1,2,3}. R = { (a,b) : a, b are co-primes}

    = { (1,1),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2) }

    (2,1), (1,2) ∈ R, but (2,2) $\notin R$. R is not transitive.
12. Removing (1,2), R' = {(1,1),(2,2)} is the identity relation.
13. A = {1,2,3}. R = {(a,b): difference between a and b is 1}.

    = {(1,2),(2,1),(2,3),(3,2)}

    R is not reflexive, not transitive. It is symmetric only.
14. A = {1,2,3,4,5,6,7}. R = {(a,b) : both a and b are either odd or even }.

    R = { (1,1),(1,3),(1,5),(1,7),(3,3),(3,5),(3,7),(5,5),(5,7)
    (1,1),(3,1),(5,1),(7,1),(3,3),(5,3),(7,3),(5,5),(7,5),
    (2,2),(2,4),(2,6),(4,4),(4,6),(6,6),(4,2),(6,2),(6,4)}

    In all these ordered pairs {1,3,5,7} are related to each other and {2,4,6} are related to each other. Therefore, [1] = { 1,3,5,7}
15. [2] = {2,4,6} . [NOTE : In the last two problems, the class can be

directly written if you are thorough with the ncert book.]

16. A = {1,2,3,4,5,6,7}. R = { (a,b) : b > 2a } = {(1,3),(1,4),(1,5),(1,6),(1,7),(2,5), (2,6),(2,7),(3,7)}. DO IT YOURSELF. [ANS : NOT REFLEXIVE]

17. DO IT YOURSELF. [ANS : NOT SYMMETRIC]

18. DO IT YOURSELF. [ANS : TRANSITIVE]

19. A = {1,2,3,4,5,6,7}. R = {(a,b) : a is a divisor of b}

R = { (1,1),(1,2),(1,3),(1,4),(1,5),(1,6),(1,7),(2,2),(2,4),(2,6),(3,3),(3,6),(4,4),(5,5),(6,6),(7,7)}. R is reflexive.

20. DO IT YOURSELF. [ANS : NOT SYMMETRIC]

21. DO IT YOURSELF. [ANS : TRANSITIVE]

22. DO IT YOURSELF. [ANS : USE THE FORMULA 2^{n^2-n}].

23. DO IT YOURSELF. [ANS : USE THE FORMULA $2^{\frac{n(n+1)}{2}}$]

24 . The function f(x) = e^x is not onto, because e^x is always positive. There is no x in the domain for which , say – 1 is the image.

25. The function is not one – one, because, f(1) = f(- 1). That is., two different elements of the domain have the same image.

26. The function is not onto, because the co-domain contains zero , Positive and negative reals and for all x in the domain, the image is always a positive real. For example, there is no x in domain for which – 1 is the image.

27. (3, ∞)

28. One well – known possibility that domain is R and co-domain is R.

[NOTE : IN QUESTIONS 29 AND 30 THE DOMAIN ELEMENTS ARE NOT INDIVIDUAL NUMBERS, BUT ORDERED PAIRS OF NATURAL NUMBERS ., (1,1),(1,2(1,3),. . . ,(2,1),(2,2) , . . .]

29. The function f is not one-one. For, f(1,2) = 2 = f(2,1).

    Two different elements of the domain have the same image.

30. The function f is onto, because (as the co-domain is N) every natural number can be factored in natural numbers. For example, 1 = 1 x 1, 2 = 1 x 2 = 2 x 1 , 10 = 1 x 10 = 10 x 1 = 2 x 5 = 5 x 2 , etc.,

31. The function y = $f(x) = x^{1/3} \Rightarrow$ x = y^3 ; For every y ∈ R(co-domain), y^3 = x exists in R(domain). Therefore, f is onto.

32. f(x) = f(y) ⇒ $x^{1/3} = y^{1/3} \Rightarrow x = y$. So, f is one - one.

33. DO IT YOURSELF. [ANS : 6]

34. DO IT YOURSELF. [ANS : 6}

35. The function is not onto, because for negative reals in the co-domain, say for – 1 ∈ $R(co - domain)$, there is no x in domain for which f(x) = x^2 = - 1.

36. The function is not one-one. For, f(x) = f(y) ⇒ $x^2 = y^2$ ⇒ x = $\pm y$.

    So, f(x) = x^2 = f(- x).

37. The function $f: (-\infty, 0] \to [0, \infty)$ defined by f(x) = x^2 is onto.

38. The function $f: (-\infty, 0] \to [0, \infty)$ defined by f(x) = x^2 is one – one.

39. The function $f: [0, \infty) \to [0, \infty)$ defined by f(x) = x^2 is onto.

40. The function $f: [0, \infty) \to [0, \infty)$ defined by f(x) = x^2 is one – one.

41. The function f(x) = x^3 is onto. The function f(x) = $4x^3 + 5$ is a Transformation of f(x) = x^3. So, it is onto.

42. The function f(x) = x^3 is one-one. The function f(x) = $4x^3 + 5$ is a Transformation of f(x) = x^3. So, it is one-one.

43. The function is not one – one.

44. The function is one-one.

45. The function is not onto.

46. $f(x) = \frac{x}{x+1} = y\,, say$. Then, x = xy + y ; x – xy = y ; x = $\frac{y}{1-y}$.

Therefore, x is defined for all y ∈ $R(co - domain)$, except y = 1.

The range is R – {1}.

47. When the domain is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, co-domain is [- 1 , 1], the function is one-one and onto.

48. When the domain is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, co-domain is R, the function is one-one but not onto. [NOTE : Other co-domains are also possible].

49. When the domain is $[0,2\pi]$, co-domain is [- 1 , 1], the function is not one-one but onto.

50. The domain is R and co-domain is R.

51. DO IT YOURSELF.

52. DO IT YOURSELF.

END OF CHAPTER I

// K.MANI , SALEM, TAMILNADU //

II. INVERSE TRIGONOMETRIC FUNCTIONS

VERY SHORT ANSWER TYPE

1. If $y = \sin^{-1} x$, $x \in [-1,1]$ state the range of principal value of y.
2. If $y = \cos^{-1} x$, $x \in [-1,1]$ state the range of principal value of y.
3. If $y = \tan^{-1} x$, $x \in R$ state the range of principal value of y.
4. If $y = \cot^{-1} x$, $x \in R$ state the range of principal value of y.
5. If $y = \sec^{-1} x$, $x \in R - (-1,1)$ state the range of principal value of y.
6. If $y = \text{cosec}^{-1} x$, $x \in R - (-1,1)$ state the range of principal value of y.
7. If $y = \sin^{-1} x$, for what values of x, y is defined ?.
8. If $y = \cos^{-1} x$, for what values of x, y is defined ?.
9. If $y = \tan^{-1} x$, for what values of x, y is defined ?.
10. If $y = \cot^{-1} x$, for what values of x, y is defined ?.
11. If $y = \sec^{-1} x$, for what value of x, y is defined ?
12. If $y = \text{cosec}^{-1} x$, for what values of x, y is defined?.
13. State the principal value of $\sin^{-1} 0$.
14. State the principal value of $\sin^{-1} 1$.
15. State the principal value of $\sin^{-1}(-1)$.
16. State the principal value of $\cos^{-1} 0$.
17. State the principal value of $\cos^{-1} 1$.
18. State the principal value of $\cos^{-1}(-1)$.
19. State the principal value of $\tan^{-1} 0$.
20. State the principal value of $\tan^{-1} 1$.
21. State the principal value of $\tan^{-1}(-1)$.
22. State the principal value of $\cot^{-1} 0$.

23. State the principal value of $\cot^{-1} 1$.

24. State the principal value of $\cot^{-1}(-1)$.

25. State the principal value of $\sec^{-1} 1$.

26. State the principal value of $\sec^{-1}(-1)$.

27. State the principal value of $\text{cosec}^{-1} 1$.

28. State the principal value of $\text{cosec}^{-1}(-1)$.

29. Find the value of $\sin\left(\tan^{-1}\left(\frac{1}{2}\right)\right)$.

30. Find the value of $\tan\left(\sec^{-1}\left(\sqrt{3}\right)\right)$.

31. Find the value of $\sin\left(\cos^{-1}\frac{1}{2}\right)$.

32. Find the value of $\sin\left(\frac{\pi}{4} - 3\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$.

33. Find the value of $\cos\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$.

34. Evaluate : $\sin\left[\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right) + \tan^{-1}\left(\sqrt{3}\right)\right]$

35. Evaluate : $\tan\left(\frac{3\pi}{2} - \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right)$.

36. Evaluate : $\sin\left(\frac{4}{3}\tan^{-1}(-1)\right)$.

37. If $f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$, *find* $f(0)$.

38. If $f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$, *find* $f(1)$.

39. If $f(x) = \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$, *find* $f(-1)$.

40. If $\sin^{-1} x + \cos^{-1}\left(-\frac{1}{2}\right) = \pi$, find x.

41. If $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + 2\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\cot^{-1} x$, find x.

42. If $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \sec^{-1}(x) = \pi$, find x.

43. If $\tan^{-1} x + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{2}$, find x.

44. If $x \in \left[0, \frac{1}{\sqrt{2}}\right]$, find which is greater : $\sin^{-1} x$ (or) $\cos^{-1} x$.

45. If $\sin^{-1} x = \frac{\pi}{4}$, *what is* $\cos^{-1}(-x)$?

46. If $\sin^{-1}\left(x - \frac{1}{2}\right) = \cot^{-1}(1)$, find x.

47. If $\sin^{-1}(x - 1) = \frac{\pi}{2} + \tan^{-1}(-1)$, find x.

48. If $\sin(\sin^{-1} x + \sin^{-1} 1) = \frac{1}{2}$, find x.

49. Evaluate : $\sin\left(\frac{3\pi}{2} - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$.

50. Evaluate : $tan\left(\frac{\pi}{2} - \sec^{-1}(-\sqrt{2})\right)$.

51. Evaluate : $\sin^{-1}\left(\sin\frac{5\pi}{6}\right)$.

52. Evaluate : $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

53. Evaluate : $\cos^{-1}\left(\cos\frac{5\pi}{4}\right)$.

54. Evaluate : $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$.

55. Evaluate : $\cos\left(\tan^{-1}\left(\frac{3}{4}\right)\right)$.

56. Write in simplified form : $\sin(\cot^{-1} x)$.

57. Write in simplified form : $\sec(\sin^{-1} x)$.

58. If $\sin(\sin^{-1} x + \sin^{-1} 1) = x$, find x.

59. Find the domain of $\sin^{-1}\left(\frac{x}{2}\right)$.

60. Find the domain of $\cos^{-1}(x^2)$.

61. Find the domain: $\cos^{-1}(x + 1)$.

62. Find the domain: $\sin^{-1}(1 - x)$.

63. Find the domain of $\sin^{-1}(x^2 - 1)$.

64. Evaluate : $\sin(\tan^{-1} 2)$.

65. Evaluate : $\sin^{-1}(\cos(\tan^{-1}(-1)))$.

66. Evaluate : $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right) + \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$.

67. Find the value : $\sin(\tan^{-1}\sqrt{3}) + \cos(\cot^{-1}\sqrt{3})$.

68. Find the value : $\tan\left(\sin^{-1}\frac{1}{\sqrt{3}}\right) + \cot\left(\cos^{-1}\frac{1}{\sqrt{3}}\right)$.

69. Prove that $(\sin^{-1} x + \cos^{-1} x)^2 = \frac{\pi^2}{4}$, $x \in [-1,1]$.

II. INVERSE TRIGONOMETRIC FUNCTIONS

SOLUTIONS /ANSWERS

1. $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. 2. $y \in [0, \pi]$. 3. $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

4. $y \in (0, \pi)$. 5. $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$. 6. $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$.

7. $y = \sin^{-1} x$ *is defined when* $x \in [-1,1]$.

8. $y = \cos^{-1} x$ *is defined when* $x \in [-1,1]$.

9. $y = \tan^{-1} x$ *is defined when* $x \in R$.

10. $y = \cot^{-1} x$ *is defined when* $x \in R$.

11. $y = \sec^{-1} x$ *is defined when* $x \in R - (-1,1)$.

12. $y = \text{cosec}^{-1} x$ *is defined when* $x \in R - (-1,1)$.

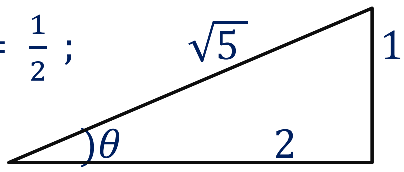
13. 0 14. $\frac{\pi}{2}$ 15. $-\frac{\pi}{2}$ 16. $\frac{\pi}{2}$

17. 0 18. π 19. 0 20. $\frac{\pi}{4}$

21. $-\frac{\pi}{4}$ 22. $\frac{\pi}{2}$ 23. $\frac{\pi}{4}$ 24. $\frac{3\pi}{4}$

25. 0 26. π 27. $\frac{\pi}{2}$ 28. $-\frac{\pi}{2}$

29. $\sin\left(\tan^{-1}\left(\frac{1}{2}\right)\right)$. Let $\theta = \tan^{-1}\left(\frac{1}{2}\right) \Rightarrow \tan\theta = \frac{1}{2}$;



Hypotenuse $= \sqrt{5}$; $\therefore \sin\theta = \frac{1}{\sqrt{5}}$.

30. $\tan(\sec^{-1}(\sqrt{3}))$. Let $\theta = \sec^{-1}(\sqrt{3}) \Rightarrow \sec\theta = \sqrt{3}$;

Opposite side $= \sqrt{2}$; $\therefore \tan\theta = \sqrt{2}$.

31. $\sin\left(\cos^{-1}\frac{1}{2}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$.

32. $\sin\left(\frac{\pi}{4} - 3\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right) = \sin\left(\frac{\pi}{4} - 3\cdot\frac{\pi}{3}\right) = \sin\left(-\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

33. $\cos\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right) = \cos\left(\frac{\pi}{2} - \left(-\frac{\pi}{3}\right)\right) = -\frac{\sqrt{3}}{2}$.

34. $\sin\left[\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)+\tan^{-1}(\sqrt{3})\right] = \sin\left(\frac{2\pi}{3}+\frac{\pi}{3}\right) = 0.$

35. $\tan\left(\frac{3\pi}{2} - \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right)\right) = \tan\left(\frac{3\pi}{2} - \frac{3\pi}{4}\right) = -1.$

36. $\sin\left(\frac{4}{3}\tan^{-1}(-1)\right) = \sin\left(\frac{4}{3}\times-\frac{\pi}{4}\right) = -\frac{\sqrt{3}}{2}.$

37. $\mathrm{f(x)} = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$

$f(0) = \sin^{-1}0 + \cos^{-1}0 + \tan^{-1}0 = \frac{\pi}{2}.$

38. If $\mathrm{f(x)} = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$,

$f(1) = \sin^{-1}1 + \cos^{-1}1 + \tan^{-1}1 = \frac{\pi}{2} + 0 + \frac{\pi}{4} = \frac{3\pi}{4}$

39. If $\mathrm{f(x)} = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$,

$f(-1) = \sin^{-1}(-1) + \cos^{-1}(-1) + \tan^{-1}(-1),$

$= \left(-\frac{\pi}{2}\right) + \pi + \left(-\frac{\pi}{4}\right) = \frac{\pi}{4}.$

40. $\sin^{-1}x + \cos^{-1}\left(-\frac{1}{2}\right) = \pi \Rightarrow \sin^{-1}x = \pi - \frac{2\pi}{3} \Rightarrow x = \frac{\sqrt{3}}{2}.$

41. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + 2\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\cot^{-1}x.$

$-\frac{\pi}{3} + 2\left(-\frac{\pi}{6}\right) = -\cot^{-1}x \Rightarrow x = -\frac{1}{\sqrt{3}}.$

42. $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) + \sec^{-1}(x) = \pi$; $\sec^{-1}(x) = \pi - \frac{\pi}{4} \Rightarrow x = -\sqrt{2}.$

43. $\tan^{-1}x + \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{2} \Rightarrow \tan^{-1}x = \frac{\pi}{2} - \frac{\pi}{6} \Rightarrow x = \sqrt{3}.$

44. $\mathrm{x} \in \left[0, \frac{1}{\sqrt{2}}\right]$, $\sin^{-1}x$ (or) $\cos^{-1}x$. Substituting some sample test values for x from the given interval, it can be found that $\cos^{-1}x$ is greater. ***OR*** It is easier to find from the graphs(DELETED PORTION)

45. $\sin^{-1}x = \frac{\pi}{4} \Rightarrow x = \frac{1}{\sqrt{2}} \therefore \cos^{-1}(-x) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}.$

46. $\sin^{-1}\left(x - \frac{1}{2}\right) = \cot^{-1}(1) \Rightarrow \sin^{-1}\left(x - \frac{1}{2}\right) = \frac{\pi}{4}$

$\Rightarrow x - \frac{1}{2} = \frac{1}{\sqrt{2}} \Rightarrow \ x = \frac{1}{2} + \frac{1}{\sqrt{2}}$.

47. $\sin^{-1}(x-1) = \frac{\pi}{2} - \frac{\pi}{4} \Rightarrow x - 1 = \frac{1}{\sqrt{2}} \ \Rightarrow x = 1 + \frac{1}{\sqrt{2}}$.

48. $\sin(\sin^{-1} x + \sin^{-1} 1) = \frac{1}{2}\,; \ \Rightarrow \ \sin\left(\sin^{-1} x + \frac{\pi}{2}\right) = \frac{1}{2}$

$\Rightarrow \ \sin^{-1} x + \frac{\pi}{2} = \frac{\pi}{6} \ \ \Rightarrow \sin^{-1} x = -\frac{\pi}{3} \Rightarrow x = -\frac{\sqrt{3}}{2}$.

49. $\sin\left(\frac{3\pi}{2} - \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\right) = \sin\left(\frac{3\pi}{2} - \frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$.

50. $tan\left(\frac{\pi}{2} - \sec^{-1}(-\sqrt{2}\,)\right) = \tan\left(\frac{\pi}{2} - \frac{3\pi}{4}\right) = -1$.

51. $\sin^{-1}\left(\sin\frac{5\pi}{6}\right) = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$.

The principal value for $\sin^{-1}$ lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

52. $\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \tan^{-1}(-1) = -\frac{\pi}{4}$.

The principal value for $\tan^{-1}$ lies in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

53. $\cos^{-1}\left(\cos\frac{5\pi}{4}\right) = \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) = \frac{3\pi}{4}$.

The principal value for $\cos^{-1}$ lies in $[0, \pi]$.

54. $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right)$. Let $\theta = \cos^{-1}\left(\frac{3}{5}\right)$; $\cos\theta = \frac{3}{5}$. $\therefore$ *opposite side* $= 4$.

So, $\sin\left(\cos^{-1}\left(\frac{3}{5}\right)\right) = \sin\theta = \frac{4}{5}$.

55. $\cos\left(\tan^{-1}\left(\frac{3}{4}\right)\right)$. Let $\theta = \tan^{-1}\left(\frac{3}{4}\right)$; hypotenuse $= 5$.

Proceeding as above $\cos\left(\tan^{-1}\left(\frac{3}{4}\right)\right) = \frac{4}{5}$.

56. $\sin(\cot^{-1} x)$. Let $\theta = \cot^{-1}(x)$; $\cot\theta = \mathrm{x} = \frac{x}{1}$; hypotenuse $= \sqrt{x^2+1}$

$\sin(\cot^{-1} x) = \sin\theta = \frac{1}{\sqrt{x^2+1}}$.

57. $\sec(\sin^{-1} x)$. Proceed as above. Adjacent side $= \sqrt{1-x^2}$

So, $\sec(\sin^{-1} x) = \sec\theta = \frac{1}{\sqrt{1-x^2}}$.

58. $\sin(\sin^{-1} x + \sin^{-1} 1) = x.$

i.e., $\sin\left(\sin^{-1} x + \frac{\pi}{2}\right) = x$; $\cos(\sin^{-1} x) = x$;

i.e., $\sin^{-1} x = \cos^{-1} x \Rightarrow x = \frac{1}{\sqrt{2}}$.

59. the domain of $\sin^{-1}\left(\frac{x}{2}\right)$.

The domain of $\sin^{-1} x$ is $[-1, 1]$. i.e., $-1 \le x \le 1$.

i.e., $\sin^{-1} x$ is meaningful only when $-1 \le x \le 1$.

$\sin^{-1}\left(\frac{x}{2}\right)$ is meaningful only when $-1 \le \frac{x}{2} \le 1$, $i.e.$, $-2 \le x \le 2$.

60. domain of $\cos^{-1}(x^2)$. $\cos^{-1} x$ is meaningful only when $-1 \le x \le 1$.

Proceeding as above, $-1 \le x^2 \le 1$; $0 \le x^2 \le 1 \Rightarrow -1 \le x \le 1$.

61. Please try. Ans : [- 2 , 0]

62. Please try. Ans : [0 , 2]

63. Please try yourself. Ans : $[-\sqrt{2}, \sqrt{2}]$

64. $\sin(\tan^{-1} 2)$, $\theta = \tan^{-1} 2$; $\tan\theta = 2 = \frac{2}{1}$; hypotenuse $= \sqrt{5}$

So, $\sin(\tan^{-1} 2) = \sin\theta = 2/\sqrt{5}$.

65. $\sin^{-1}(\cos(\tan^{-1}(-1))) = \sin^{-1}\left(\cos\left(-\frac{\pi}{4}\right)\right) = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$.

66. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(-\frac{1}{2}\right) + \cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)$

$= \left(-\frac{\pi}{3}\right) + \left(\frac{2\pi}{3}\right) + \left(\frac{2\pi}{3}\right) = \pi.$

67. $\sin(\tan^{-1}\sqrt{3}) + \cos(\cot^{-1}\sqrt{3}) = \sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{6}\right) = \sqrt{3}$.

68. Please try yourself. ANS : $\sqrt{2}$

69. $(\sin^{-1} x + \cos^{-1} x)^2 = \frac{\pi^2}{4}$, $x \in [-1,1]$.

Let $\sin^{-1} x = \theta$; then $\sin\theta = x$; $\cos^{-1} x = \frac{\pi}{2} - \theta$

Sub, in LHS, you get RHS.

END OF CHAPTER 2

III. MATRICES

VERY SHORT ANSWER TYPE

1. If A = $\begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$, find the condition on the values of a,b and c for which A is invertible.

2. Construct a 2 x 2 matrix A = [a_{ij}] whose elements are given by

   $a_{ij} = \begin{cases} \frac{i}{j}, & i \neq j \\ 0, & i = j \end{cases}$.

3. Construct a 2 x 2 matrix A = [a_{ij}] whose elements are given by

   $a_{ij} = \begin{cases} \frac{i+j}{i-j}, & i \neq j \\ ij, & i = j \end{cases}$.

4. Write the number of all possible matrices of order 2 x 2 with entries either 0 or 1 ?

5. Write the number of all possible matrices of order 2 x 2 with entries either – 1 or 1 ?

6. Write the number of all possible matrices of order 2 x 2 with entries – 1 or 0 or 1 ?

7. Find the number of matrices of order 2 x 2 with entries 0 or 1 which are invertible.

8. Find the number of matrices of order 2 x 2 with entries – 1 or 1 which are invertible.

9. If A = $\begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ is invertible and $A^{-1} = A$, find a,b and c.

10. A square matrix A of order 3 x 3 which is invertible is such that $A^2 = A$. Prove that A = I.

11. Suppose A is a 2 x 1 matrix and B is a 1 x 2 matrix. Prove that AB is Not invertible.

12. If $\begin{bmatrix} 3 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [6]$, what can be the values of x and y ?

13. If $\begin{bmatrix} 3 & 4 & 7 \end{bmatrix}\begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix} A = \begin{bmatrix} 5 & 10 \end{bmatrix}$, find the matrix A.

14. If $\begin{bmatrix} x & y \end{bmatrix} \cdot I = \begin{bmatrix} 1 & 2 \end{bmatrix}$, find x,y.

15. If a matrix A of order 2 x 2 is both symmetric and skew-symmetric, find |A|.

16. If A^{-1} is the inverse of a matrix $A_{3 \times 3}$, what is the inverse of 5A ?

17. Find a 2 x 2 matrix A for which $A^2 = A$.

18. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, find θ if $A = A^T$.

19. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, find θ if $A = A^{-1}$.

20. If the matrix $A = \begin{bmatrix} x & 0 \\ 0 & -y \end{bmatrix}$ is such that $A^2 = A$, find x , y.

21. If the matrix $A = \begin{bmatrix} 0 & x \\ y & 0 \end{bmatrix}$ is such that $A^2 = A$, find x , y.

22. If $A = \begin{bmatrix} p^2 & 0 & 0 \\ 0 & 3q & 0 \\ 0 & 0 & 9 \end{bmatrix}$ is a scalar matrix, find the values of p , q.

23. If $A = \begin{bmatrix} 0 & p & 3 \\ -2 & 0 & 2 \\ p+c & r & 0 \end{bmatrix}$ is a skew-symmetric matrix , find p,r,c .

24. Find x and y if $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

25. Let A be a 3 x 5 matrix, B a 5 x 2 matrix, C a 3 x 4 matrix, D a 4 x 2 matrix and E a 4 x 5 matrix. Write the order of each of the following matrices, if exists.

1) $3(EB) + 5D$ 2) $CD - 2(CE)B$ 3) $2(EB)+DA$ 4) $AB + CD$.

26. Let A be a 2 x 2 matrix, B a 2 x 2 matrix, C a 2 x 3 matrix, D a 3 x 2 matrix and E a 3 x 1 matrix. Write the order of each of the following matrices, if exists.

1) A^2C 2) $B^3 + 3CD$ 3) $DA - 2(DB)$ 4) $4A(BC)$ 5) $C + 3D$.

27. Simplify the matrix expression : $(A - B)(A + B) - (A + B)^2$.

28. Simplify the matrix expression : $A(A + B) - B(A + B)$.

29. Let P be a 3 x 2 matrix, Q a 2 x 1 matrix, R a 1 x 3 matrix, S a 3 x 1 matrix and T a 3 x 3 matrix. Write the order of each of the following matrices, if exists.

1) $4TSPQ + 3\,PQ$ 2) $QRSR + QR$.

30. If $A = \begin{bmatrix} -4 & 8 \\ -2 & 4 \end{bmatrix}$, find A^2.

31. If A be a m x n matrix, B a n x r matrix, C a r x q matrix , find the number of elements in ABC.

32. Let A be a 4 x 1 matrix, B a 2 x 3 matrix, C a 2 x 4 matrix, D a 1 x 3 matrix. Write the order of each of the following matrices, if exists.

1) ADB^T 2) $C^TB - 5AD$ 3) $4CA - (CA)^2$

4) $(ADB^TC)^2 + I$ 5) $(B^TC)^T - AD$.

33. If $A = [a_{ij}]$ and $B = [b_{ij}]$ are matrices of order 2 x 1 and 2 x 1 respectively, defined by $a_{ij} = i + j$ and $b_{ij} = i - j$, find AB^T.

34. If A is a skew-symmetric matrix, find whether $A^{-1}$is also skew-symmetric.

35. If $M = ABC$, where A,B and C are invertible matrices, what is M^{-1} ?

36. If A and B are symmetric matrices, find whether ABA is also symmetric.

37. Find whether $A(BC)^T$ and AB^TC^T are equal matrices.

38. Find whether $A(B + C)^T$ and $AB^T + AC^T$ are equal matrices.

39. If $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$, find A^2.

40. If $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, find A^2.

41. If $A = \begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$, find A^2.

42. If $A = \begin{bmatrix} 2 & c & c \\ c & c & c \\ 8 & 7 & c \end{bmatrix}$, find "c" for which A is invertible.

43. Let $A = \begin{bmatrix} a & b & b \\ a & a & b \\ a & a & a \end{bmatrix}$, find the conditions on the values of a and b for which A is invertible.

44. If A is a symmetric and invertible matrix, find whether A^{-1} is also symmetric.

45. If A and B are invertible matrices such that $A^2 = I$ and $B^2 = I$, prove That $(AB)^{-1} = BA$.

46. If the inverse of A^2 is B, show that the inverse of A is AB.

47. Let the product $M = ABC$ of three square matrices of same order be Invertible and A,B,C are invertible. Find an expression for B^{-1}.

III. MATRICES

[CASE STUDY BASED MCQS]

I. THE WORD TRANSFORMATION SIMPLY MEANS CHANGE.

We can use matrices to effect some transformations. For example,

1. ROTATION : The ***rotation matrix*** $R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

can be used for rotation about the origin of a point (or some figure in

2-dimensional plane) through an angle θ in the anticlockwise direction if θ is positive and in the clockwise direction if θ is negative.

E.g., The point , say P(3,2) is rotated about the origin through $\frac{\pi}{2}$.

The image of P is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

2. REFLECTION : The transformation T which changes the point P(x,y) to the point P'(x , - y) is called a reflection. The point P' is the mirror image of P in the x – axis. The ***reflection matrix*** *is* $T = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

[NOTE : While dealing with transformations, points with coordinates (x,y) are written as a 2 x 1 column matrix].

There are many more transformations and compositions of these.

The matrix transformations are used in computer graphics, robotics, Architecture, biology etc.,

Now, answer the following questions :

1. The image of the point (2,1) under rotation through an angle of π in anticlockwise direction is

   a) (- 2, 1) b) (- 2 , - 1) c) (- 1 , 2) d) (2, - 1)

2. The image of the point (2,0) under rotation through an angle $\pi/6$ In clockwise direction is

   a) (- $\sqrt{3}$, 1) b) ($\sqrt{3}$, - 1) c) (2 , - 1) d) (1, - 1)

3. The image of the point (- 3 , 1) under reflection T is

   a) (3, - 1) b) (- 2 , - 1) c) (- 1 , 3) d) (- 3, - 1)

4. If rotation R($\frac{\pi}{2}$) is followed by reflection , the image of (0, - 5) is

   a) (- 5, 0) b) (- 5 , 5) c) (0 , 5) d) (5, 0)

5. Two consecutive rotations, in order R_1 ($\frac{3\pi}{2}$) followed by R_2 ($\frac{\pi}{4}$)

take the point (- 2 , - 4) to the point

a) $(-3\sqrt{2}, \sqrt{2})$ b) $(-3\sqrt{2}, -\sqrt{2})$

c) $(-3\sqrt{2}, -2\sqrt{2})$ d) $(-3\sqrt{2}, -2\sqrt{2})$.

II. A manufacturer produces three products x,y, z which he sells in three markets.The annual sales are indicated below :

MARKET	PRODUCTS		
	x	y	z
I	16,000	2000	18,000
II	6,000	20,000	8,000
III	12,000	11,000	10,000

The selling prices per unit of x, y and z are Rs. 4, Rs. 3 and Re. 2 respectively. The cost prices per unit of x, y and z are Rs. 3, Rs. 2 and Re. 1 respectively.

Let A denote the annual sales matrix, S denote the selling price matrix and C denote the cost price matrix.

$$A = \begin{bmatrix} 16{,}000 & 2{,}000 & 18{,}000 \\ 6{,}000 & 20{,}000 & 8{,}000 \\ 12{,}000 & 11{,}000 & 10{,}000 \end{bmatrix};\ S = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix};\ C = [3\ \ 2\ \ 1].$$

Now, answer the following questions :

1. The grand total revenue is

a) 106,000 b) 100,000 c) 101,000 d)307,000

2. The gross profit is

a) 194,000 b) 307,000 c) 103,000 d) none of these

3. The revenue from market II is

a) 106,000 b) 101,000 c) 194,000 d) 100,000

4. The cost of producing the product x is given by

a) $\begin{bmatrix}16{,}000\\6{,}000\\12{,}000\end{bmatrix}\cdot C$

b) $\begin{bmatrix}16{,}000\\6{,}000\\12{,}000\end{bmatrix}\cdot C^T$

c) $C\cdot\begin{bmatrix}16{,}000\\6{,}000\\12{,}000\end{bmatrix}$

d) $C^T\cdot\begin{bmatrix}16{,}000\\6{,}000\\12{,}000\end{bmatrix}$

5. The highest individual market revenue is given by

a) $[16{,}000 \quad 2{,}000 \quad 18{,}000]\cdot s^T$

b) $S\cdot[16{,}000 \quad 2{,}000 \quad 18{,}000]$

c) $S^T\cdot[16{,}000 \quad 2{,}000 \quad 18{,}000]$

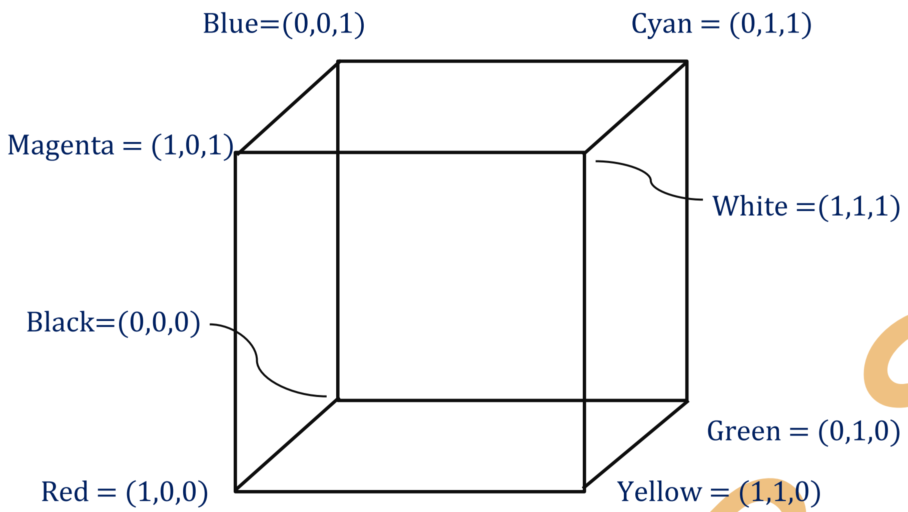
d) $[16{,}000 \quad 2{,}000 \quad 18{,}000]\cdot S$

III. The colors for computer monitors are often described using ordered triples. One model called the RGB system, uses red, green and blue to generate all colors. The figure describes the relationships of these colors in the system. Red is (1,0,0), green is (0,1,0) and blue is (0,0,1). Since equal amounts of red and green combine to form yellow, yellow is represented by(1,1,0).

Similarly, magenta(a deep reddish purple) is a mixture of blue and red and is represented by (1,0,1). Cyan is (0,1,1), since it is a mixture of blue and green.

Another color model uses cyan, magenta and yellow. Referred to as the CMY model, it is used in the four-color printing process for textbooks. In this system cyan is (1,0,0), magenta is (0,1,0) and yellow is (0,0,1). In the CMY model, red is created by mixing magenta and yellow. Thus red is (0,1,1) in this system.

To convert ordered triples in the RGB model to ordered triples in the CMY model, we can use the following matrix equation. In both of these systems, color intensities vary between 0 and 1.



MATRIX EQUATION

$$\begin{bmatrix} C \\ M \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} R \\ G \\ B \end{bmatrix}$$

Now, answer the following questions :

1. If (0.631,1,0.933) is aquamarine in the RGB model, then the mixture in The CMY model is

a) (0.369,1,0.067)
b) (0.369,0,0.067)
c) (0.369,1,0)
d) (0.631,0,0.933)

2. The equation which gives RGB values corresponding to the given CMY values is

a) $\begin{bmatrix} C \\ M \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} R \\ G \\ B \end{bmatrix}$

b) $\begin{bmatrix} R \\ G \\ B \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} C \\ M \\ Y \end{bmatrix}$

c) $\begin{bmatrix} C \\ M \\ Y \end{bmatrix} = \begin{bmatrix} R \\ G \\ B \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

d) $\begin{bmatrix} C \\ M \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} R \\ G \\ B \end{bmatrix}$

3. If (0.012,0,0.597) is cream color in the CMY model, then the mixture in RGB model is

a) (0.988,1,0.403) b) (0.012,1,0.597)

c) (0.988,0,0.597) d) none of these

4. To get magenta, the mixture must have

a) blue = 0 b) green = 0

c) red = 0 d) none of these.

5. If the color white is a mixture, which of the following color is missing?

a) red b) green

c) blue d) none of these.

III. MATRICES

SOLUTIONS/PARTIAL SOLUTIONS/ANSWERS

1. | A | = 2abc. If A is to be invertible, $|A| \neq 0$.

So, none of a,b,c can be 0.

2. $A = \begin{bmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{bmatrix}$ 3. $A = \begin{bmatrix} 1 & -3 \\ 3 & 4 \end{bmatrix}$

4. Each entry can be filled in two ways , either by 0 or by 1.

So, number of matrices = 2 x 2 x 2 x 2 = 16.

5. 16.

6. 3 x 3 x 3 x 3 = 81.

7. *DO IT YOURSELF (D I Y)*

8. *DO IT YOURSELF (D I Y)*

9. $A^{-1} = \begin{bmatrix} 0 & {}^{1}/_{c} \\ {}^{1}/_{b} & {}^{-a}/_{bc} \end{bmatrix}$; $A^{-1} = A \Rightarrow a = 0;\ bc = 1.$

So, a = 0 and b and c can be any two real numbers with bc = 1.

10. A is invertible. So, A^{-1} exists. $A^2 = A \Rightarrow (AA)\, A^{-1} = AA^{-1}$

$\Rightarrow A(AA^{-1}) = I$; $\Rightarrow AI = I$; $\Rightarrow A = I$.

11. *DO IT YOURSELF.* [ANS:AB is always singular.So, A^{-1} does not exist].

12. Multiplying matrices in LHS , $[\,3x + 0y\,] = [6] \Rightarrow x = 2$.

y can be any real number.

13. $[\,3 \quad 4 \quad 7\,]\begin{bmatrix}5\\1\\-2\end{bmatrix}A = [\,5 \quad 10\,]$; LHS has orders 1 x 3, 3 x 1 , A.

RHS has 1 x 2. So, order of A must be 1 x 2.

LHS = $[\,5\,][\,x \quad y\,] = [\,5x \quad 5y\,]$; So, x = 1, y = 2.

14. $[\,x \quad y\,]\, I = [\,1 \quad 2\,]$; $[\,x \quad y\,]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = [\,1 \quad 2\,]$;

$[\,x + 0 \quad 0 + y\,] = [\,1 \quad 2\,]$; So x = 1; y = 2.

15. A is symmetric $\Rightarrow A^T = A$; A is skew-symmetric $\Rightarrow A^T = -A$;

So, $A = -A \Rightarrow A = 0$; $|A| = 0$.

16. Use $(k\,A)^{-1} = \frac{1}{k}A^{-1}$; where k ≠ 0 is any real number.

17. *DO IT YOURSELF (D I Y).* [ANS : One possibility is A = I].

18. $A = \begin{bmatrix}\cos\theta & \sin\theta\\-\sin\theta & \cos\theta\end{bmatrix}$; $A^T = \begin{bmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{bmatrix}$;

$A = A^T \Rightarrow \sin\theta = -\sin\theta$; $2\sin\theta = 0$; $\theta = n\pi, n \in Z$.

19. *DO IT YOURSELF (D I Y).* [ANS : Proceed as in Qn.18. $\theta = n\pi, n \in Z$].

20. $A = \begin{bmatrix}x & 0\\0 & -y\end{bmatrix}$; $A^2 = \begin{bmatrix}x^2 & 0\\0 & y^2\end{bmatrix}$; $A^2 = A \Rightarrow x^2 = x$; $y^2 = -y$.

Solving , x = 0, 1 ; y = 0 , - 1.

21. *DO IT YOURSELF (D I Y)* . [ANS : x = 0 ; y = 0].

22. A scalar matrix is a diagonal matrix in which all the diagonal elements are equal. So, $p^2 = 9$; $3q = 9$; $\Rightarrow p = \pm 3$; $q = 3$.

23. In a skew-symmetric matrix, $a_{ij} = -a_{ji}$. So , p = 2 ; r = - 2 ;

p + c = - 3 ; c = - 5.

24. *DO IT YOURSELF (D I Y).*[ANS : $x = \cos\theta$; $y = -\sin\theta$].

25. 1) 4 x 2 2) 3 x 2 3) does not exist 4) 3 x 2

26. *DO IT YOURSELF (D I Y)* [ANS : 1) 2 x 3 2) 2 x 2 3) 3 x 2 4) 2 x 3 5) does not exist].

27. (A - B)(A + B) – (A + B)² = AA + AB – BA – BB – (A + B)(A + B)

= AA + AB – BA – BB – [AA + AB + BA + BB]

= AA + AB – BA – BB – AA - AB - BA - BB

= - 2 BA – 2 BB [OR] = - 2B(A + B).

You can try the other way also.

28. A(A + B) – B(A + B) = AA + AB – BA – BB = A^2 + AB – BA – B^2.

[OR] = (A - B)(A + B).

29. *DO IT YOURSELF (D I Y)* ; [ANS:1) TSPQ + 3PR does not exist, 2) 2].

30. *DO IT YOURSELF (D I Y)* ; [ANS : $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$]

31. *DO IT YOURSELF (D I Y)* ; [ANS : mq elements]

32. *DO IT YOURSELF.* [ANS : 4 X 2, 4 X 3, does not exist, 4 x 4, 4 x 3].

33. *DO IT YOURSELF (D I Y)* [ANS : $\begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$].

34. A is symmetric. So, $A^T = A$. Let A^{-1} be the inverse of A.

Now, $(A^{-1})^T = (A^T)^{-1} = A^{-1}$; So, $(A^{-1})^T = A^{-1}$; A^{-1} also symmetric.

35. $M^{-1} = C^{-1}B^{-1}A^{-1}$, Using reversal law for inverses.

36. *DO IT YOURSELF.* [ANS : Yes].

37. *DO IT YOURSELF.* [ANS : Not equal].

38. *DO IT YOURSELF.* [ANS : Equal].

39. *DO IT YOURSELF.* [*ANS*: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$]. 40. *DO IT YOURSELF.* [*ANS*: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$].

41. *DO IT YOURSELF.* [ANS : $A^2 = A$].

42. If c = 0 or 2 or 7, |A| = 0. So, For A to be invertible, $c \in R - \{0,2,7\}$

43. $a \neq 0, a \neq b$.   44. *DO IT YOURSELF.* [ANS : Yes, symmetric].

45. $A^2 = I$; i.e., AA = I ; post-multiplying by A^{-1}, $A = A^{-1}$.

Similarly , $B^{-1} = B$. Now, use inverse reversal law.

46. *DO IT YOURSELF (D I Y)*

[ANS : $(A^2)^{-1} = B;\ A^{-1}A^{-1} = B$; Premultiply both sides by A].

47. *DO IT YOURSELF (D I Y)* [ANS : $B^{-1} = CM^{-1}A$].

III. MATRICES - CASE STUDIES

SOLUTIONS/PARTIAL SOLUTIONS/ANSWERS

I. 1. b	2. b	3. d	4. d	5. b
II. 1. d	2. c	3. d	4. c	5. d
III. 1. b	2. d	3. a	4. b	5. d

END OF CHAPTER 3

// K.MANI , SALEM, TAMILNADU //

IV. DETERMINANTS

VERY SHORT ANSWER TYPE

1. If $A = \begin{bmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$, $find\ the\ value\ of\ |A - I|$.

2. Solve : $\begin{vmatrix} x+2 & 3x \\ 3 & x+2 \end{vmatrix} = 0.$

3. If A is a 2 x 2 matrix all whose entries are 1, find the value of |2(A – I)|.

4. If $\begin{vmatrix} 0 & 1 & sec\ \alpha \\ tan\ \alpha & -sec\ \alpha & tan\ \alpha \\ 1 & 0 & 1 \end{vmatrix} = 1, find\ the\ value\ of\ \alpha.$

5. Find x if $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$.

6. If A = $\begin{bmatrix} 4 & -2 \\ 2 & 1 \end{bmatrix}$, $find$ |5A|.

7. If $\begin{vmatrix} x & 3 & 3 \\ 3 & 3 & x \\ 2 & 3 & 3 \end{vmatrix} = 0, find\ x.$

8. Find x if $\begin{bmatrix} 4 & 3 & 5 \\ 3 & -2 & 7 \\ 10 & -1 & x \end{bmatrix}$ is singular.

9. If $\begin{vmatrix} 1 & a & b \\ 1 & b & c \\ 1 & c & a \end{vmatrix} = K(a^2 + b^2 + c^2)$, $find\ k.$

10. If the points (p , x), (x , q) and (x , x) are collinear, find the value of x.

11. If A is an invertible matrix of order and |A| = 5, what is $|A^{-1}|$?

12. If A is an invertible matrix of order 3 x 3 and |A| = 5, what is |kA|?

13. If A is a skew-symmetric matrix of order 3 x 3, then what is A(adj A) ?

14. For what value(s) of "k", the matrix $\begin{bmatrix} 1 & k & k \\ k & 1 & k \\ k & k & 1 \end{bmatrix}$ $is\ non-singular$?

15. Evaluate without expanding : $\begin{vmatrix} 112 & 129 & 104 \\ 67 & 78 & 62 \\ 99 & 114 & 92 \end{vmatrix}$.

16. Let f(x) = $\begin{vmatrix} 1 - sin^2x & 1 + sin^2x \\ 1 - cos^2x & 1 + cos^2x \end{vmatrix}$. Evaluate f(x) and hence find $f\left(\frac{\pi}{2}\right)$.

17. Solve for x : $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+x \end{vmatrix} = 0.$

18. If $\begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix} = K\,abc, find\ k.$

19. If $\begin{bmatrix} a & 2 & y \\ x & b & 5 \\ 3 & z & c \end{bmatrix}$ is a skew-symmetric matrix, find the values of

x + y + z and |A|.

20. If A is a 2 x 2 matrix and |A| = 3, what is $|(A^T)^2|$?

21. If $\begin{vmatrix} 0 & x^2 & y^3 \\ x^3 & 0 & z^2 \\ y^2 & z^3 & 0 \end{vmatrix} = 0$, $where\ x, y\ and\ z\ are\ all\ non-zero$, find the

value of xyz.

22. If A is a 2 x 2 matrix and |A| = 3, what is $|4A^{-1}|$?

23. If A and B are 3 x 3 matrices and |A| = - 3 , |B| = 2, find $|3A^2B|$.

24. If A and B are 3 x 3 matrices and |A| = - 3 , |B| = 2, find $|2AB^{-1}|$.

25. If A and B are 3 x 3 matrices and |A| = - 3 , |B| = 2, find $||(AB^{-1})^T|$.

26. If $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = a^3 + b^3 + c^3 + Kabc$, what is K ?

27. Find the number of real solutions of $\begin{vmatrix} x & -1 & 0 \\ 0 & x & 1 \\ 1 & 0 & x \end{vmatrix} = 0.$

28. Find the value of θ if $\begin{vmatrix} \cos\theta & -\sin\theta & 1 \\ \sin\theta & \cos\theta & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0.$

29. Find whether (a + b + c) is a factor of $\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$.

30. If $0 \le x, y \le 1, what\ is\ the\ maximum\ value\ of$ $\begin{vmatrix} 1 & 1 & 1 \\ 1 & x+y & x-y \\ 1 & x-y & x+y \end{vmatrix}$?

31. If $\begin{vmatrix} ab & bc & ca \\ bc & ca & ab \\ ca & ab & bc \end{vmatrix} = 0$, find the value of ab + bc + ca.

32. Find the value of $\begin{vmatrix} cos^2 x & \cos x \sin x & -\sin x \\ \cos x \sin x & sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$.

33. Find the value of $\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$.

34. Find the value of $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$,where x,y,z are all positive Real numbers.

IV. DETERMINANTS - SOLUTIONS

1. $A = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$, $|A - I| = 11$.

2. Expanding, $x^2 + 4x + 4 - 9x = 0$; x = 1,4.

3. – 4.

4. Expansion gives , $sec^2\alpha = 1$;

$tan^2\alpha = 0$;

$\alpha = 0$ or in general $n\pi$.

5. $x = \pm 6$.

6. $|5A| = 5^2.|A| = 25 \times 8 = 200$.

7. expansion gives $x^2 - 5x + 6 = 0$; Therefore, x = 2,3.

8. x = -19.

9. LHS = $ab + bc + ca - a^2 - b^2 - c^2$; k = - 1 .

[ab + bc + ca = 0 is not needed]

10. x = p,q.

11. $|A^{-1}| = \frac{1}{|A|} = \frac{1}{5}$.

12. $|kA| = k^3|A| = 5k^3$.

13. If A is any square matrix, A(adj A) = |A|I .

The determinant of a skew-symmetric matrix of odd order is 0.

Therefore, O(zero matrix).

14. If k = 1 , - ½, the determinant value is 0.

The matrix is non-singular if, $k \in R - \{1, -\frac{1}{2}\}$

15. 6.

16. Expanding and factorizing, you get f(x) = 2($\cos^2$x – $\sin^2$x) = 2 cos 2x.

And, $f\left(\frac{\pi}{2}\right)$ = 2 cos π = - 2.

17. x = 0 , - 3.

18. K = 2.

19. If A is skew-symmetric, x = - 2, y = - 3, z = - 5 and in a skew-symmetric matrix, the main diagonal elements are zeroes.

Therefore, a = b = c = 0. So, x + y + z = - 10.

|A| = 0, see the answer to QN. 13.

20. A is a 2 x 2 matrix and |A| = 3.

$|(A^T)^2| = |(A^T)(A^T)| = |A^T||A^T| = |A||A| = 9.$

21. Expansion gives, $x^2y^2z^2(1 + xyz) = 0;$

$As\ x, y, z\ are\ non-zero,$ 1 + xyz = 0 ; xyz = - 1.

22. 4^2 x 1/3 = 16/3.

23. |A| = - 3 , |B| = 2. $|3A^2B| = 3^3\,|A^2B|$

$= 27|A^2|\,|B|$

= 27 x 9 x 2 = 486.

24. |A| = - 3 , |B| = 2 ; $|2AB^{-1}| = 2^3\,|A||B^{-1}|$ = 8 x (- 3) x ½ = - 12.

25. |A| = - 3 , |B| = 2 ; $|(AB^{-1})^T| = |(B^{-1})^T A^T|$

$= |(B^{-1})^T|\,|A^T| = |B^{-1}||A| = 1/2.(-3) = -\frac{3}{2}.$

26. Expansion gives, $\Delta = a^3 + b^3 + c^3 - 3abc$. $\therefore K = -3.$

27. Expansion gives, $x^3 - 1 = 0$. This has three roots. One root is 1 and the other two roots are complex.

28. $\Delta = 0 \Rightarrow cos\theta = 0. \quad \therefore \theta = \frac{\pi}{2}$.

$In\ general,\ \theta = (2n+1)\frac{\pi}{2}, n \in Z.$

29. Expansion gives $\Delta = a^3 + b^3 + c^3 - 3abc$.

$$= (a+b+c)(a^2+b^2+c^2-ab-bc-ca).$$

Therefore, a + b + c is a factor.

30. Expansion gives, $\Delta = 4xy - 4y$. The maximum value is 0.

31. Expansion and $\Delta = 0$ gives ,

$a^3b^3 + b^3c^3 + c^3a^3 = 3a^2b^2c^2$. Let A = ab ; B = bc ; C= ca.

Then, $A^3 + B^3 + C^3 = 3ABC \Rightarrow A + B + C = 0.\ i.e.$, ab+bc+ca = 0.

32. Expanding , we get

$cos^4x + cos^2x \cdot sin^2x - sinx(-cos^2xsinx - sin^3x)$

$= cos^4x + cos^2x \cdot sin^2x + sin^2x = 1$, on factorization.

33. It ‘s in skew-symmetric form. The determinant value is 0.

34. Expand the determinant and use the rule of logarithms,

$\log_b a \log_a b = 1$, the value of the determinant is 0.

END OF CHAPTER 4

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V. CONTINUITY AND DIFFERENTIABILITY

VERY SHORT ANSWER TYPE

1. If the function f defined by $f(x) = \begin{cases} 2x\,, & x \leq 1 \\ k\,, & x > 1 \end{cases}$ is continuous at x = 1, find k.

2. If the function f defined by $f(x) = \begin{cases} x^3, & x \leq 2 \\ ax^2\,, & x > 2 \end{cases}$ is continuous at x = 2, find a.

3. If the function f defined by $f(x) = \begin{cases} \frac{4\sin x}{x}\,, & x < 0 \\ a - 2x\,, & x \geq 0 \end{cases}$ is continuous at x = 0, find a.

4. If the function f defined by $f(x) = \begin{cases} \frac{x^2 - a^2}{x - a}\,, & x \neq a \\ 8\,, & x = a \end{cases}$ is continuous at x = a, find a.

5. Specify, if any , the points of discontinuity of the function f(x) = [x] , $-1 \leq x \leq 1$.

6. What are the points of discontinuity of f(x) = $\left[\frac{x}{2}\right]$, $-1 \leq x \leq 1$.

7. What are the points of discontinuity of f(x) = $[2x]$, $-1 \leq x \leq 1$.

8. Give an example of a function which is continuous and also differentiable on R.

9. Give an example of a function which is continuous and also differentiable on a sub-interval of R.

10. If the function $f(x) = \begin{cases} cx + 1, & x \leq 3 \\ cx^2 - 1\,, & x > 3 \end{cases}$ is continuous at x = 3, find c.

11 . Is the function $f(x) = \begin{cases} \frac{1}{3}x + 4, & x < 3 \\ 2x - 1\,, & x \geq 3 \end{cases}$ continuous at x = 3 ?

12. Is the function defined by $f(x) = \frac{1}{x^2 - 7x + 10}$ continuous at $x = 5$?

13. Specify the points of discontinuity of $f(x) = \frac{1}{x^2 - 3x + 2}$.

14. Check if the function $f(x) = \begin{cases} x, & x \neq 1 \\ 2, & x = 1 \end{cases}$ is continuous at $x = 1$.

15. If the function $f(x) = \begin{cases} 1, & x \geq 0 \\ x + a, & x < 0 \end{cases}$ is continuous at $x = 0$, find a.

16. If the function $f(x) = \begin{cases} 2(x - a), & x \geq 0 \\ x^2 + 1, & x < 0 \end{cases}$ is continuous at $x = 0$, find a.

IN QUESTIONS 17 – 51 ,FIND THE FIRST ORDER DERIVATIVE W.R.TO X:

17. $\sqrt{log\ x}$
18. $\left(\frac{x}{e}\right)^x$
19. cos(m sinx)
20. $\log_x e^x$
21. $x^{\sqrt{x}}$
22. $\log_{e^x} x$
23. $\log_{x^n} x$
24. $\sqrt{1 + \sqrt{x}}$
25. $e^{\log e^x}$
26. cos(sinx)
27. $\sqrt{x^x}$
28. $\log_{\sin x} e$
29. $\log_{e^2} \sin x$
30. $\sqrt{xe^x}$
31. $e^{2/x}$
32. $\sin^3 3x$
33. $|5x|, x \neq 0$
34. $\cos^{-1}(\sin x)$
35. $\sin^{-1} e^x$
36. $\tan^{-1} 2x$
37. $2^{\tan^{-1} x}$
38. $\cos^{-1}(\log x)$
39. $\left(\cot^{-1}\sqrt{x}\right)^2$
40. $\sin^{-1}(1 - 2x^2)$
41. $\cos^{-1}(2x^2 - 1)$
42. $\frac{\pi}{2} - \sin^{-1}(1 - x^2)$
43. $\left(\frac{1}{x}\right)^x$
44. $x^{1/x}$

45. $\left(\frac{1}{x}\right)^{1/x}$

46. $|2x - 3|$, $x \neq 3/2$

47. $|\sin x|$, $x \neq 0$

48. $|\cos x|$, $x \neq 0$

49. $\sin |x|$, $x \neq 0$

50. $\sqrt{\frac{1 - tan^2 x}{1 + tan^2 x}}$

51. $\frac{1 - cos\ 2x}{1 + cos\ 2x}$

IN QUESTIONS 52 – 71 ,FIND $dy/_{dx}$:

52. $y = \frac{1 + x}{1 + y}$

53. $x = \frac{1 - y}{1 + y}$

54. $x = y^{1/3}$

55. $\sin x + \sin y = \cos x + \cos y$

56. $\frac{1}{x} + \frac{1}{y} = 1.$

57. $a^2y^2 = x^2(a^2 - x^2)$

58. $\sqrt{x} + \sqrt{y} = \sqrt{a}$

59. $ax^2 + by^2 = c.$

60. $x^y = k$

61. $y^x = 2.$

62. $y = x^2 + 2^{logx}$

63. $y^x = x$

64. $e^x + e^y = 1$

65. $x^2y = 1$

66. $\frac{y}{x} = \frac{1 - y}{1 + x}$

67. $sin\ y = e^x$

68. $\cos y = \sqrt{x}$

69. $y = \sin(xy)$

70. $y = \cos(x + y)$

71. $y = \log_e\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)$

72. If $\tan(x + y) = x$, find dy/dx at (0,0).

73. If $x = at^2$, $y = a/t^2$, find dy/dx at t = -1/2.

74. If $x = acos^2\theta, y = bsin^2\theta$, $find\ \frac{dy}{dx}$.

75. If $x = a \cos \theta$, $y = b \sin \theta$, find dy/dx at $\theta = 3\pi/4$.

76. If $x = \cos \theta$, $y = 3 \sin \theta$ find dy/dx at $\theta = 3\pi/2$.

77. If $x = tan^2\theta, y = sec^2\theta , find \ \frac{dy}{dx}$.

78. If f(x) = x|x|, find f '(x).

79. If f(x) = x|x|, find whether f "(0) exists.

80. If f(x) = |x − 2|, $find\ f\ '(0)$.

81. Find f '(0) , if exists where f(x) = $\sqrt{x}$.

82. Find whether the function y = $\sqrt{sin\ x}$ is differentiable in $\left[0 , \frac{\pi}{2}\right]$?

83. Find whether the function y = $\sqrt{sin\ x}$ is differentiable in $\left(0 , \frac{\pi}{2}\right)$?

84. Find whether the function y = $\sqrt{sin\ x}$ is differentiable in $(0 , \pi)$? a

85. If the function f(x) defined by $f(x) = \begin{cases} \frac{tan\ 3x}{sin\ 2x} , & x \neq 0 \\ k , & x = 0 \end{cases}$ is continuous at x = 0, find k.

86. The function f(x) is defined by $f(x) = \begin{cases} 2 , & x \neq 0 \\ k , & x = 0 \end{cases}$.

For what real values of k the function is discontinuous at x = 0.

87. Differentiate w.r. to x : $(ae)^x$.

88. Differentiate w.r. to x : $e^{x + \log x}$

89. Differentiate w.r. to x : $\sin^2 3x$

90. Differentiate w.r. to x : $\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$.

V. CONTINUITY AND DIFFERENTIABILITY – SOLUTIONS

A FUNCTION f(x) IS SAID TO BE CONTINUOUS AT A POINT x = c IF THE LHL, RHL, f(c) ARE ALL EQUAL .

1. GIVEN : $f(x) = \begin{cases} 2x, & x \le 1 \\ k, & x > 1 \end{cases}$ is continuous at x = 1. ∴ LHL = RHL.

   LHL = $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} 2x = \lim_{h \to 0} 2(1-h) = 2$;

   RHL = $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} k = k$. Therefore k = 2.

2. $f(x) = \begin{cases} x^3, & x \le 2 \\ ax^2, & x > 2 \end{cases}$ is continuous at x = 2.

   LHL = $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^3 = \lim_{h \to 0} (2-h)^3 = 8$;

   RHL = $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax^2 = \lim_{h \to 0} a(2+h)^2 = 4a$.; 4a = 8; a = 2.

3. $f(x) = \begin{cases} \frac{4 \sin x}{x}, & x < 0 \\ a - 2x, & x \ge 0 \end{cases}$ is continuous at x = 0.

   LHL = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{4 \sin x}{x} = 4$;

   RHL = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} a - 2x = \lim_{h \to 0} a - 2(0+h) = a$; a = 4.

4. $f(x) = \begin{cases} \frac{x^2 - a^2}{x - a}, & x \ne a \\ 8, & x = a \end{cases}$ is continuous at x = a.

   LHL = $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} \frac{x^2 - a^2}{x - a} = 2a^{2-1} = 2a$;

   RHL = $\lim_{x \to a^+} f(x) = \lim_{x \to a^+} \frac{x^2 - a^2}{x - a} = 2a$; f(a) =8; 2a = 8 ; a = 4.

5. f(x) = [x] , $-1 \le x \le 1$. The function, the GIF(greatest integer function) is discontinuous at integer points. The point of discontinuity is 0,1.

   Read the definition of continuity in a closed interval [a,b].

6. ***DO IT YOURSELF.*** [ANS : 0].

7. ***DO IT YOURSELF.*** [ANS : - ½, 0, ½,1].

8. sin x, cos x, e^x , e^{-x} are continuous and differentiable on R.

9. y = $\sqrt{25 - x^2}$ is continuous and differentiable on a subinterval of R.

10. $f(x) = \begin{cases} cx + 1, & x \le 3 \\ cx^2 - 1, & x > 3 \end{cases}$ is continuous at x = 3.

LHL = $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} cx + 1 = \lim_{h \to 0} c(3-h) + 1 = 3c + 1;$

RHL = $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} cx^2 - 1 = \lim_{h \to 0} c(3+h)^2 - 1 = 9c - 1;$

3c +1 = 9c – 1 ; c = 1/3.

11. $f(x) = \begin{cases} \frac{1}{3}x + 4, & x < 3 \\ 2x - 1, & x \ge 3 \end{cases}$

LHL = $\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{1}{3} x + 4 = \lim_{h \to 0} \frac{1}{3}(3-h) + 4 = 5;$

RHL = $\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2x - 1 = \lim_{h \to 0} 2(3+h) - 1 = 5;$

f(3) = 3 = LHL = RHL . The function is continuous.

12. f(x) = $\frac{1}{x^2 - 7x + 10} = \frac{1}{(x-2)(x-5)}$; f(x) is not defined at x = 5.

∴ f(x) is not continuous at x = 5.

13. ***DO IT YOURSELF.*** [ANS : x = 1,2].

14. $f(x) = \begin{cases} x, & x \ne 1 \\ 2, & x = 1 \end{cases}$. $\lim_{x \to 1} f(x) = \lim_{x \to 1} x = 1.$ But, f(1) = 2.

$\lim_{x \to 1} f(x) \ne f(1).$ ∴ f(x) is not continuous at x = 1.

15. $f(x) = \begin{cases} 1, & x \ge 0 \\ x + a, & x < 0 \end{cases}$ is continuous at x = 0. f(0)=1.

LHL = $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x + a = \lim_{h \to 0} (0-h) + a = a;$

RHL = $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$; f(x) is continuous. So, a = 1.

16. $f(x) = \begin{cases} 2(x-a), & x \ge 0 \\ x^2 + 1, & x < 0 \end{cases}$ is continuous at x = 0.

LHL = $\lim_{x \to 0^-} f(x) = \lim_{x\to 0^-} x^2 + 1 = \lim_{h\to 0} (0-h)^2 + 1 = 1$;

RHL = $\lim_{x \to 0^+} f(x) = \lim_{x\to 0^+} 2(x - a) = \lim_{h \to 0} 2[(0+h) - a] = -2a$;

LHL = RHL . $-2a = 1$; a = - ½.

17) $\frac{1}{2\sqrt{\log x}} \cdot \frac{1}{x} = \frac{1}{2x\sqrt{\log x}}$.

18) *Let* $y = \left(\frac{x}{e}\right)^x$; Taking log , log y = x log$\left(\frac{x}{e}\right)$ = x(logx – loge);

log y = x(log x − 1); $\frac{1}{y}y' = x.\frac{1}{x} + (\log x - 1)\cdot 1 = \log x$;

y' = y$(\log x) = \left(\frac{x}{e}\right)^x (\log x)$

19) y = cos(msin x); y' = $-\sin(m\sin x) \cdot m\cos x$

$= -m\cos x\sin(m\sin x)$.

20) y = $\log_x e^x = \frac{\log e^x}{\log x} = \frac{x}{\log x}$; $y' = \frac{\log x \cdot 1 - x\cdot {}^1/_x}{(\log x)^2} = \frac{\log x - 1}{(\log x)^2}$.

21) y = $x^{\sqrt{x}}$; $\log y = \sqrt{x} \cdot \log x$; $\frac{1}{y}y' = \sqrt{x}\,\frac{1}{x} + \log x \cdot \frac{1}{2\sqrt{x}}$;

Therefore, y' = y$[\frac{1}{\sqrt{x}} + \frac{\log x}{2\sqrt{x}}] = \frac{y}{2\sqrt{x}}\{2 + \log x\}$.

22) y = $\log_{e^x} x = \frac{\log x}{\log e^x} = \frac{\log x}{x\log e} = \frac{\log x}{x}$;

$y' = \frac{x\cdot 1/x - \log x\cdot 1}{x^2} = \frac{1 - \log x}{x^2}$.

23) y = $\log_{x^n} x = \frac{\log x}{\log x^n} = \frac{\log x}{n\log x} = \frac{1}{n}$; $y' = 0$.

24) y = $\sqrt{1 + \sqrt{x}}$; $y' = \frac{1}{2\sqrt{1+\sqrt{x}}} \times \left(0 + \frac{1}{2\sqrt{x}}\right) = \frac{1}{4\sqrt{x}\,\sqrt{1+\sqrt{x}}}$.

25) y = $e^{\log e^x} = e^{x\log e} = e^{x\cdot 1} = e^x; y' = e^x$.

[OR] Use $e^{\log f(x)} = f(x)$.

26) y = cos(sin x); y' = $-\sin(\sin x) \cdot \cos x$.

27) y = $\sqrt{x^x} = x^{x/2}$; $\log y = \frac{x}{2} \cdot \log x$;

$$\frac{1}{y}y' = \frac{1}{2}\left(x \cdot \frac{1}{x} + logx \cdot 1\right)$$

$$= \frac{1}{2}(1 + logx)\ ;\ y' = \frac{y}{2}(1 + logx).$$

28) $y = \log_{\sin x} e = \dfrac{1}{\log_e \sin x}$;

$$y' = \frac{log\ sinx \cdot 0 - 1 \cdot cot\ x}{(log\ sinx)^2} = \frac{-\ cot\ x}{(log\ sinx)^2}.$$

29) $y = \log_{e^2} \sin x = \dfrac{log\ sin\ x}{log\ e^2} = \dfrac{log\ sin\ x}{2}\ ;\ y' = \dfrac{1}{2}cot\ x.$

30) $y = \sqrt{xe^x}\ ;\ y' = \dfrac{1}{2\sqrt{xe^x}}(xe^x + e^x \cdot 1)$

$$= \frac{e^x(x+1)}{2\sqrt{xe^x}} = \frac{(x+1)}{2}\sqrt{\frac{e^x}{x}}.$$

31) $y = e^{2/x}\ ;\ y' = \left(\dfrac{-2}{x^2}\right) e^{2/x}.$

32) $y = \sin^3 3x\ ;\ y' = 3 \sin^2 3x.\cos 3x.3 = 9 \sin^2 3x.\cos 3x.$

33) $y = |5x|, x \neq 0\ ;\ y' = \dfrac{5x}{|5x|} \cdot 5$

34) $y = cos^{-1}(sinx)\ ;\ y' = \dfrac{-1}{\sqrt{1-(\sin x)^2}} \times cosx = -1.$

35) $y = \sin^{-1} e^x\ ;\ y' = \dfrac{1}{\sqrt{1-(e^x)^2}} \cdot e^x = \dfrac{e^x}{\sqrt{1-e^{2x}}}.$

36) $y = \tan^{-1} 2x\ ;\ y' = \dfrac{1}{1+(2x)^2} \cdot 2 = \dfrac{2}{1+4x^2}.$

37) $y = 2^{\tan^{-1} x}\ ;\ y' = e^{\tan^{-1} x} \cdot log2 \cdot \dfrac{1}{1+x^2}.$

38) $y = \cos^{-1}(log\ x)\ ;\ y' = \dfrac{-1}{\sqrt{1-(log\ x)^2}} \cdot \dfrac{1}{x} = \dfrac{-1}{x\sqrt{1-(log\ x)^2}}.$

39) $y = \left(\cot^{-1}\sqrt{x}\right)^2$;

$$y' = 2(\cot^{-1}\sqrt{x})\ \frac{-1}{1+\left(\sqrt{x}\right)^2} \cdot \frac{1}{2\sqrt{x}} = \frac{-\cot^{-1}\sqrt{x}}{\sqrt{x}\,(1+x)}.$$

40) $y = \sin^{-1}(1-2x^2)$; put $x = \sin\theta$;

$$y = \sin^{-1}(1 - 2sin^2\theta) = \sin^{-1} cos2\theta = \sin^{-1} sin\left(\frac{\pi}{2} - 2\theta\right)$$

$$= \left(\frac{\pi}{2} - 2\theta\right) = \frac{\pi}{2} - 2\sin^{-1} x \; ; \; y\,' = \frac{-2}{\sqrt{1-x^2}}.$$

41) $y = \cos^{-1}(2x^2 - 1)$; put $x = \cos\theta$;

$$y = \cos^{-1}(2cos^2\theta - 1) = \cos^{-1} cos\, 2\theta$$

$$= 2\theta = 2\cos^{-1} x; y' = \frac{-1}{\sqrt{1-x^2}}$$

42) $y = \frac{\pi}{2} - \sin^{-1}(1 - x^2) = \cos^{-1}(1 - x^2)$;

$$y' = \frac{-1}{\sqrt{1-(1-x^2)^2}} \cdot (-2x) = \frac{2x}{\sqrt{1-(1-x^2)^2}}.$$

43) $y = \left(\frac{1}{x}\right)^x$; $log\, y = xlog\left(\frac{1}{x}\right)$

$$= x(log1 - log\, x) = -x\, log\, x;$$

$$y\,' = -(x \cdot 1/x + log\, x \cdot 1) = -(1 + log\, x).$$

44) $y = x^{1/x}$; $\log y = \frac{1}{x} log\, x = \frac{log\, x}{x}$; $\frac{1}{y} y' = \frac{x \cdot 1/x - log\, x \cdot 1}{x^2}$

$$\frac{y\,'}{y} = \frac{1 - log\, x}{x^2} \; ; \; y' = y\left(\frac{1 - log\, x}{x^2}\right).$$

45) $y = \left(\frac{1}{x}\right)^{1/x}$; $log\, y = \frac{1}{x}\log\left(\frac{1}{x}\right) = \frac{1}{x}(log\, 1 - log\, x) = \frac{-log\, x}{x}$;

$$\frac{y\,'}{y} = \frac{-(1 - log\, x)}{x^2} \; ; \; y' = y\left(\frac{log\, x - 1}{x^2}\right).$$

46) $y = |2x - 3|$, $x \neq 3/2$; $y\,' = \frac{2x-3}{|2x-3|} \cdot 2 = \frac{4x-6}{|2x-3|}$.

47) ***DO IT YOURSELF.*** [ANS : $\frac{\sin x}{|\sin x|} . \cos x$]

48) ***DO IT YOURSELF.***

49) ***DO IT YOURSELF.***

50) $y = \sqrt{\frac{1 - tan^2x}{1 + tan^2x}} = \sqrt{cos\, 2x}$;

$$y' = \frac{1}{2\sqrt{cos\, 2x}} \cdot (-sin\, 2x) \cdot 2 = \frac{-\, sin\, 2x}{\sqrt{cos\, 2x}}.$$

51) $y = \frac{1 - cos\ 6x}{1 + cos\ 6x} = \frac{1 - cos\ 2(3x)}{1 + cos\ 2(3x)} = tan^2 3x$;

$y' = 2tan3x \cdot sec^2 3x.3 = 6tan3x \cdot sec^2 3x$

52) $y = \frac{1 + x}{1 + y}$; $y + y^2 = 1 + x$; $y' + 2yy' = 1$; $y' = \frac{1}{1 + 2y}$.

53) $x = \frac{1 - y}{1 + y}$; $x + xy = 1 - y$;

$1 + x.y' + y.1 = -y'$; $xy' + y' = -1 - y$; $y' = \frac{-(1 + y)}{x + 1}$.

54) $x^2 = y^3$; $2x = 3y^2y'$; $y' = \frac{2x}{3y^2} = \frac{2x}{3x^{4/3}} = \frac{2}{3x^{1/3}}$.

55) $\sin x + \sin y = \cos x + \cos y$;

$\cos x + \cos y\ y' = -sin\ x - sin\ y\ y'$;

$y'(\cos y + \sin y) = -sin\ x - cos\ x$; $y' = -\frac{sin\ x + cos\ x}{cos\ y + sin\ y}$.

56) $\frac{1}{x} + \frac{1}{y} = 1$; $\frac{-1}{x^2} + \frac{-1}{y^2}y' = 0$; $\frac{-1}{y^2}y' = \frac{1}{x^2}$; $y' = -\frac{y^2}{x^2}$.

57) $a^2y^2 = x^2(a^2 - x^2)$; $a^2y^2 = a^2x^2 - x^4$;

$a^2.2yy' = a^2.2x - 4x^3$; $y' = \frac{a^2x - 2x^3}{a^2y}$.

58) $\sqrt{x} + \sqrt{y} = \sqrt{a}$; $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0$; $y' = -\frac{\sqrt{y}}{\sqrt{x}}$.

59) $ax^2 + by^2 = c$; $2ax + 2by.y' = 0$; $y' = -\frac{ax}{by}$.

60) $x^y = k$; $y \log x = \log k$; $y \cdot \frac{1}{x} + logx \cdot y' = 0$;

$y' = -\frac{y}{x\ log\ x}$.

61) $y^x = 2$; $x\ log\ y = log\ 2$;

$x\frac{1}{y}y' + log\ y \cdot 1 = 0$; $y' = -\frac{y\ log\ y}{x}$.

62) $y = x^2 + 2^{logx}$; $y' = 2x + 2^{logx}.log\ 2 \cdot \frac{1}{x}$;

63) $y^x = x$; $x\ log\ y = log\ x$;

$\log y = \frac{\log x}{x}; \frac{1}{y}\, y' = \frac{x\cdot 1/x - \log x \cdot 1}{x^2}$

$\frac{y'}{y} = \frac{1-\log x}{x^2}\ ;\ y' = y\left(\frac{1-\log x}{x^2}\right).$

64) $e^x + e^y = 1; e^x + e^y \cdot y' = 0; y' = -\frac{e^x}{e^y}.$

65) $x^2y = 1\ ;\ x^2y' + y\cdot 2x = 0\ ;\ y' = -\frac{2y}{x}.$

66) $\frac{y}{x} = \frac{1-y}{1+x}$; y + xy = x − xy ; 2xy = x − y ;

2(xy' +y.1) = 1− y' ; 2xy' + y' = 1 − 2y ; y' = $\frac{1-2y}{1+2x}$.

67) $\sin y = e^x\ ;\ \cos y\, .y' = e^x\ ;\ y' = \frac{e^x}{\sqrt{1-e^{2x}}}.$

68) cos y = $\sqrt{x}$; $-\sin y \cdot y' = \frac{1}{2\sqrt{x}}\ ;\ y' = \frac{-\sin y}{2\sqrt{x}}.$

69) y = sin(xy) ; y' = cos(xy)(x.y' + y.1) ;

y' = x.y'cos(xy) + y.cos(xy) ;

y' − x.y'cos(xy) = y.cos(xy) ; y' = $\frac{\text{y.cos(xy)}}{1-\text{x.cos(xy)}}$.

70) y = cos(x + y) ; y' = $-\sin(x+y)(1+y')$;

y' = $-\sin(x+y) - y'\sin(x+y)$;

y' +y'sin(x + y) = $-\sin(x+y)$; y' = $\frac{-\sin(x+y)}{1+\sin(x+y)}$.

71) y = $\log_e\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) = \log_e e^x = x\ ;\ y' = 1.$

72) tan(x + y) = x ; sec²(x + y)(1 + y') = 1 ; 1 + y' = cos²(x + y) ;

y' = cos²(x + y) − 1 ; Therefore, dy/dx at (0,0) = 0.

73) x = at², y = a/t² . $\frac{dx}{dt} = 2at\ ;\ \frac{dy}{dt} = \frac{-2a}{t^3}$;

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2a/t^3}{2at} = -\frac{1}{t^4}.\ \therefore\ \frac{dy}{dx}\ at\ t = -1/2\ is\ -16.$

74) $x = a\cos^2\theta, y = b\sin^2\theta;$

$$\frac{dx}{d\theta} = 2acos\theta(-sin\theta)\,;\ \frac{dy}{d\theta} = 2bsin\theta cos\theta\,;$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2bsin\theta cos\theta}{2acos\theta(-sin\theta)} = -\frac{b}{a}.$$

75) x = a cos θ , y = b sin θ ; $\frac{dx}{d\theta} = a(-sin\theta)\,;\ \frac{dy}{d\theta} = bcos\theta\,;$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{bcos\theta}{a(-sin\theta)} = -\frac{b}{a}cot\theta.\ At\ \theta = 3\pi/4\,,\ \frac{dy}{dx} = \frac{b}{a}.$$

76) x = cos θ , y = 3 sin θ ; $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3\cos\theta}{(-sin\theta)} = -3cot\theta\,;$

At $\theta = 3\pi/2$, dy/dx = 0.

77) $x = tan^2\theta, y = sec^2\theta;\quad \frac{dx}{d\theta} = 2\tan\theta(sec^2\theta)\,;$

$$\frac{dy}{d\theta} = 2sec\theta sec\theta tan\theta\,;\quad \therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2sec\theta sec\theta tan\theta}{2\tan\theta(sec^2\theta)} = 1\,.$$

78) f(x) = x|x| = $\begin{cases} -x^2\ , x < 0 \\ x^2\ , x \geq 0 \end{cases}$

f ’(x) = – $2x$ $for\ x < 0\ and\ 2x\ for\ x \geq 0$.

79) ***DO IT YOURSELF.*** [ANS : f ‘(x) = 2|x|, f “(0) does not exist].

80) f(x) = |x – 2| ; $f\,'(x) = \frac{x-2}{|x-2|}.1\,;\ f\,'(0) = -1.$

81) ***DO IT YOURSELF.*** [ANS : Does not exist].

82) y = $\sqrt{sin\ x}\,;\frac{dy}{dx} = \frac{\cos x}{2\sqrt{sin\ x}}$. y is not differentiable in $\left[0\,,\frac{\pi}{2}\right]$,

since dy/dx does not exist at x = 0.

83) ***DO IT YOURSELF.*** [ANS : Yes, differentiable].

84) ***DO IT YOURSELF.*** [ANS : Yes, differentiable].

85) If the function f(x) defined by $f(x) = \begin{cases} \frac{\tan 3x}{\sin 2x}\ , & x \neq 0 \\ k\ , & x = 0 \end{cases}$

f(x) is continuous at x = 0. $\therefore \lim_{x\to 0} f(x) = f(0).$

Now, $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\tan 3x}{\sin 2x} = \lim_{x \to 0} \frac{\frac{\tan 3x}{x}}{\frac{\sin 2x}{x}} = \frac{3}{2}$; $So,\ k = \frac{3}{2}$.

86) The function f(x) is defined by $f(x) = \begin{cases} 2\ ,\ x \neq 0 \\ k\ ,\ x = 0 \end{cases}$.

$\lim_{x \to 0} f(x) = 2$; f(0) = k ; If k = 2, then the function is continuous.

∴ For the function to be discontinuous , k can be any real number other than 2.

87) DO IT YOURSELF. [ANS : $(ae)^x(\log a + 1)$].

88) DO IT YOURSELF. [ANS : $xe^x + e^x$].

89) DO IT YOURSELF. [ANS : 6 sin 3x cos 3x = 3 sin 6x].

90) DO IT YOURSELF. [ANS : $\frac{1}{2} sec^2\left(\frac{\pi}{4} + \frac{x}{2}\right)$].

END OF CHAPTER 5

// K.MANI , SALEM, TAMILNADU //

VI. APPLICATIONS OF DIFFERENTIATION

VERY SHORT ANSWER TYPE

1. Find the maximum value of f(x) = $4 - |x|$.
2. Find the minimum value of f(x) = $|2x + 3|$.
3. Find the minimum value of f(x) = $|2x - 1|$.
4. Find the minimum value of f(x) = $2|x|$.
5. Find the absolute maximum value of f(x) = $3x^2$, $[-1, 2]$.
6. Find the absolute minimum value of f(x) = $3x^2$, $[-1, 2]$.
7. Find the absolute maximum value of f(x) = $2x - 3$ on $[0,2]$.
8. Find the absolute minimum of $f(x) = \begin{cases} 2x + 2, & 0 \leq x < 1 \\ 4x^2, & 1 \leq x \leq 3 \end{cases}$.
9. Find the maximum value of f(x) = $|\sin x|$, $x \in [0.2\pi]$.
10. Find the maximum value of f(x) = $\sin x$, $x \in [\pi, 2\pi]$.
11. Find the maximum value of f(x) = $-|x + 2| + 3$.
12. Find the maximum value of f(x) = $-|x - 4| - 5$.
13. Find the maximum value of f(x) = $1 - x^2$.
14. Find the minimum value of f(x) = $x^2 - 8$.
15. Find the maximum value of f(x) = $\sin x - 3$.
16. Find the minimum value of f(x) = $\sin x + 2$.
17. Find the maximum value of f(x) = $\cos x + 4$.
18. Find the minimum value of f(x) = $\cos x - 5$.
19. Find the maximum value of f(x) = $2\sin x - 3$.
20. Find the minimum value of f(x) = $4\sin x + 1$.
21. Find the interval in which f(x) = e^x is increasing.

22. Find the interval in which $f(x) = e^{-x}$ is decreasing.
23. Find the interval in which $f(x) = \log x$ is increasing.
24. Find the interval in which $f(x) = x + 2$ is strictly increasing.
25. Find the interval in which $f(x) = 6 - 2x$ is strictly decreasing.
26. Find the interval in which $f(x) = x^2 + 5$ is strictly increasing.
27. Find the interval in which $f(x) = -x^2$ is strictly increasing.
28. Find the interval in which $f(x) = x^2$ is strictly decreasing.
29. Find the interval in which $f(x) = -x^2 + 1$ is strictly decreasing .
30. Find the interval in which $f(x) = e^{-x^2}$ is strictly increasing.
31. If $f(x) = \cos x$ is strictly decreasing on $(0, a)$, find the value of "a".
32. Find the interval in which $f(x) = 2x^2 + 2x$ is strictly increasing.
33. Does the function $f(x) = \tan x$, $[0,\pi]$ have an absolute maximum or absolute minimum ?
34. Find the slope of the normal to the curve $y^2 = 4ax$, at the origin.
35. Find the equation of tangent to the curve $y = \sin x$ at the origin.
36. Find the equation of normal to the curve $y = \sin x$ at the origin.
37. Find the equation of tangent to the curve $y = \cos x$ at $\left(\frac{3\pi}{2}, 0\right)$.
38. Find the angle which the tangent to $y = e^x$ makes with x-axis at the point where the curve cuts y-axis.
39. Find the slope of the tangent to $y = e^x$ drawn at the point where the curve cuts y-axis.
40. Find the slope of the normal to $y = e^x$ drawn at the point where the curve cuts y-axis.
41. Find the equation of the tangent to $y = e^x$ drawn at the point where the curve cuts y-axis.

42. Find the equation of the normal to $y = e^x$ drawn at the point where the curve cuts y-axis.

43. Find the slope of the tangent to $y = e^{-x}$ drawn at the point where the curve cuts y-axis.

44. Find the slope of the tangent to $y = \log x$ drawn at the point where the curve cuts x-axis.

45. Find the number of tangents having slope – 1 that can be drawn to the curve $y = \frac{1}{x-3}$.

46. Find the slope of the normal to the curve $x^2 = 4y$ at the origin.

47. Find the equation of normal to the curve $x^2 = 4y$ at the origin.

48. Find the equation of tangent to the curve $x^2 = 4y$ at the origin.

49. Find the slope of normal to the curve $y = \cos x$ at the point $x = \frac{\pi}{2}$.

50. Find the slope of tangent to the curve $y = be^{-x/a}$ at the point $x = 0$.

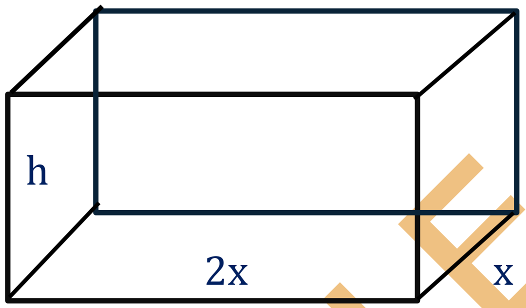
51. Find the slope of tangent to the curve $x = at^2, y = 2at$ at $t = 1$.

52. Find the absolute maximum and minimum of $f(x) = 24, x \in [4,12]$.

VI. APPLICATIONS OF DIFFERENTIATION

[CASE STUDY BASED MCQS]

I. A manufacturer of packaging boxes Manufactures boxes in the shape Of a cuboid with box 2x cm by x cm. The total outer surface of the box is to be 300 cm². See the adjoining diagram.



Now, answer the following questions:

1. The height h of the box in terms of x is

   a) $\frac{150}{x} - \frac{2x}{3}$ b) $\frac{50}{x} - \frac{2x}{3}$ c) $\frac{50}{x} + \frac{2x}{3}$ d) $\frac{50}{x} - \frac{2x^2}{3}$

2. The expression for the volume V of the box in terms x alone is

   a) $100x^2 - 4x^3$ b) $100x - 4x^3$

   c) $100x - \frac{4}{3}x^3$ d) $100 - \frac{4}{3}x^3$

3. For the volume of the box V to be largest, the value of x must be

   a) 10 cm   b) 7 cm   c) 5 cm   d) $\frac{20}{3}$ cm

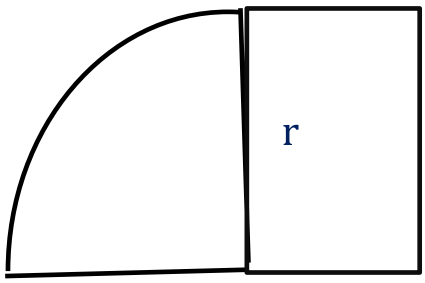
4. The maximum volume is

   a) 1000 cm³   b) 250 cm³   c) $\frac{500}{3}$ cm³   d) $\frac{1000}{3}$ cm³

5. The height h of the box when the volume is maximum is

   a) $\frac{10}{3}$ cm   b) 7 cm   c) 5 cm   d) $\frac{20}{3}$ cm

II. The cross-section of an object has the shape of a quarter circle of radius r adjoining a rectangle of width x and height r, as shown in the diagram.



The perimeter and area of the cross-section are P and A respectively. Study the diagram and answer the following questions :

1. The expression for the area A of cross-section in terms of r and x is

a) $\frac{\pi r^2}{4} + 2rx$ b) $\frac{\pi r^2}{4} + rx$ c) $\frac{\pi r^2}{2} + rx$ d) $\frac{\pi r^2}{4} + x^2$

2. The expression for the Perimeter P in terms of r and x is

a) $2x + 2r + \frac{\pi r^2}{2}$ $b)\ 2x + 2r + \frac{\pi r}{2}$

c) $2x + r + \frac{\pi r}{2}$ d) $2x + 3r + \frac{\pi r}{2}$

Treat the perimeter P of the cross-section as fixed(constant).

3. The function representing the area A in terms of P and r is(where P is constant and r is the variable)

a) $\frac{rP}{2} - 2r^2$ b) $\frac{rP}{2} - r^2$ c) $\frac{rP}{2} - 2P^2$ d) $\frac{rP}{2} - P^2$

4. The value of x in terms of r for which the area A of cross-section is maximum is

a) $\frac{4r + \pi r}{4}$ b) $\frac{4r - \pi r}{4}$ c) $\frac{4\pi - \pi r}{4}$ d) $\frac{4r - \pi r}{2}$

5. The maximum value for the area A is

a) r^2 b) $2r^2$ c) $3r^2$ d) $r^2/4$

III. The costs of a firm which makes common food item are of two kinds :

Fixed costs (plant, office expenses) ; Rs. 2000per week;
Production costs(materials, labour); Rs.20 for each item made;
Market research suggests that, if they price the item at Rs. 30 per item, they will sell 500 items per week, but that at Rs.55 per item they will sell none at all; and between these two values the graph of sales against price is a straight line.

They can price at Rs. x per item ($30 \leq x \leq 55$).

Assume that all the produced items in a week are just sold out.

Now, answer the following questions :

1. The expression for the weekly sales(number of items sold) is

a) 1100 – 20x b) 1100 + 20x

c) 2000 + 20x d) 100 + 20x

2. The expression for the weekly costs is

a) 22000 + 400x b) 22000 – 400x

c) 24000 – 400x d) 22000 – 400x

3. The expression for the weekly revenue is

a) x(1100 – 20x) b) x(1100 + 20x)

c) x(2000 + 20x) d) x(1000 - 20x)

4. The function for the weekly profit Rs. P is

a) $P(x) = 20x^2 + 1500x - 24000$

b) $P(x) = - 20x^2 + 1500x - 24000$

c) $P(x) = - 20x^2 + 1500x - 22000$

d) $P(x) = - 20x^2 + 1500x + 24000$

5. For maximum profit, the price per item(x) should be (in Rs.)

a) 30 b) 35 c) 37.50 d) 32.50

IV. A patient is administered an initial amount of injection of 100 cc of a drug(medicine) and it is monitored. After the injection, the amount A(t) of drug in the bloodstream decreases with time t, measured in hours. Suppose that under certain conditions, A(t) is given by $A(t) = \frac{A(0)}{1+t^2}$,

Where A(0) is the initial amount of the medication.

Now, answer the following questions :

1. The derivative of the function A(t) is given by

a) $\frac{100t}{1+t^2}$ b) $\frac{-200t}{(1+t^2)^2}$ c) $\frac{200t}{(1+t^2)^2}$ d) $\frac{-100t}{1+t^2}$

2. The time(in hours) when the drug is maximum is when t =

a) 2 b) 1 c) 0 d) 10

3. The curve depicting the amount of drug in the body is

a) increasing b) decreasing

c) strictly increasing d) strictly decreasing

4. During which of the following time intervals, the assimilation of the drug by the body is fastest ?

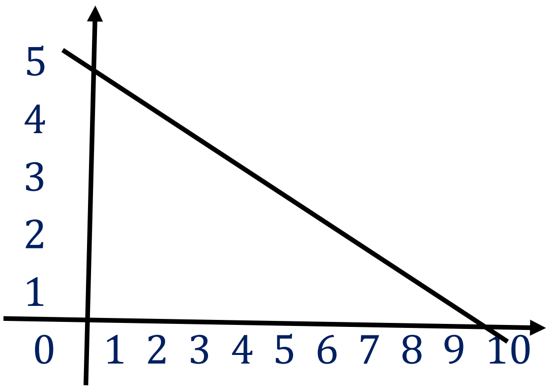
a) (1,2) b) (4,5) c) (0,1) d) none of these

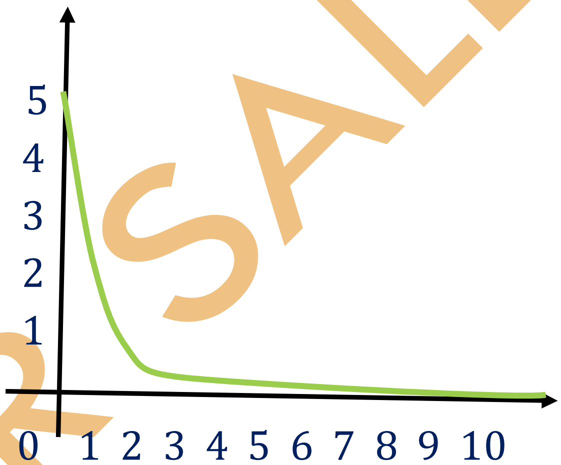
5. If the time t, in hours is taken along the x-axis and the amount of drug remaining(in cc) along y-axis then which of the following represents the graph of A(t) ?

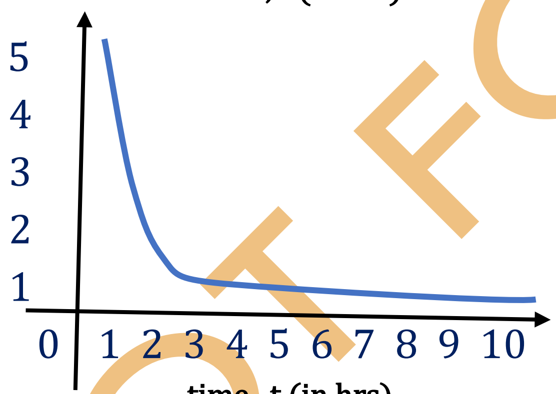
[HINT : Plot the points when t = 0, 1, 2, 7, 10].

[CHOOSE THE ONE WHICH IS VERY CLOSE IN SHAPE !!!]

SCALE : x – axis : 1 cm = 1 unit ; y – axis : 1 cm = 20 units

a) 

b) 

c) 

d) none of these

V. A clothing firm determines that in order to sell x suits, the price per unit must be $p = 150 - 0.5x$. Further, it also determines that the total cost of producing x suits is given by $C(x) = 4000 + 0.25x^2$.

Now, answer the following questions :

1. The expression for the total revenue R(x) is

a) $150 – 0.5x^2$ b) $150x – 0.5x^2$

c) $4150 – 0.25x^2$ d) $4000x + 0.25x^2$

2. The expression for the total profit P(x) is

a) $0.75x^2 - 150x + 4000$ b) $-0.75x^2 + 150x - 4000$

c) $0.75x^2 - 150x - 4000$ d) $-0.75x^2 - 150x + 4000$

3. The number of suits the company must produce and sell in order to maximize profit is

a) 100 b) 150 c) 75 d) 50

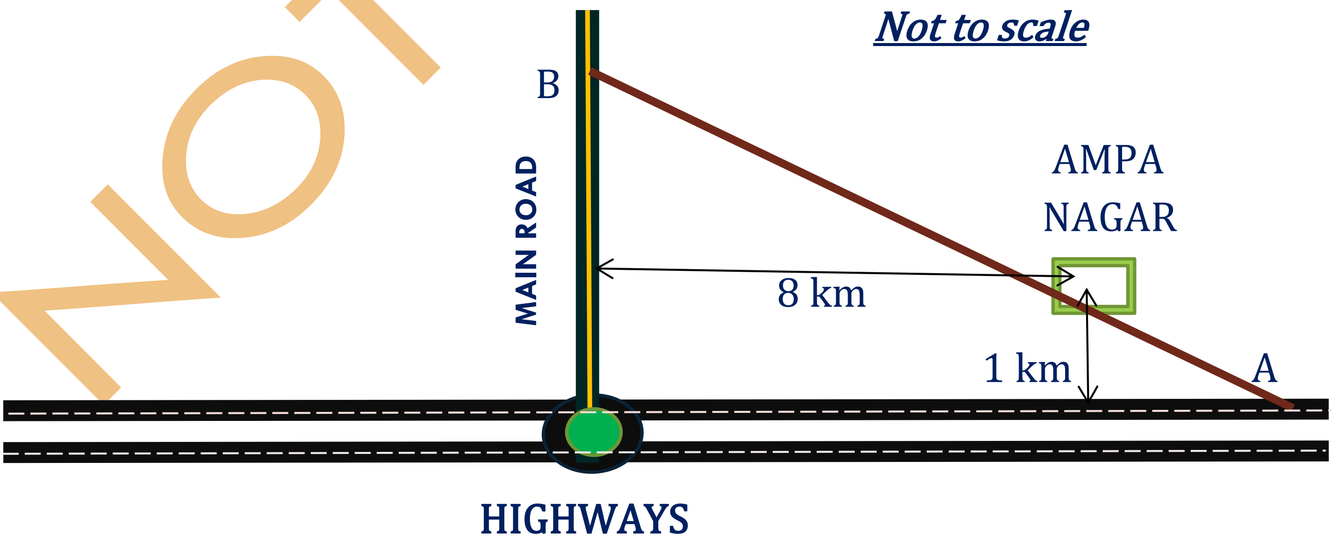
4. The maximum profit is

a) 3500 b) 10000 c) 7500 d) 11500

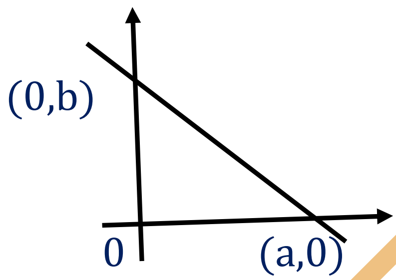
5. The price per unit(ONE SUIT) that must be charged in order to maximize profit is

a) 150 b) 100 c) 125 d) 75

VI. The residents of AMPA nagar need a road diagonally connecting the main road and the highways. The location is as specified in the figure and they propose the shortest route. You, a senior secondary student of AMPA school is asked to calculate using calculus , the route of shortest length.



Knowing that the shortest route is along a straight line you start with the equation of the line $\frac{x}{a} + \frac{y}{b} = 1.$



Now, answer the following questions :

1. The relation between a and b is

   a) $a = \frac{b}{b-1}$ b) $a = \frac{8b}{b-1}$ c) $a = \frac{b}{8b-1}$ d) $b = \frac{a}{8-a}$

2. If f = D^2 , where D is the distance between A and B then function f is

   a) $f = \frac{b^2}{(b-1)^2} + b^2$ b) $f = \frac{b^2}{64(b-1)^2} + b^2$

   c) $f = \frac{64b^2}{(b-1)^2} + b^2$ d) $f = \frac{a^2}{(8-a)^2} + a^2$

3. The critical point is

   a) b = 6   b) b = 4   c) a = 12   d) b = 5

4. The condition for minimum at a critical point is

   a) f ' changes from negative to positive   b) f '' = 0

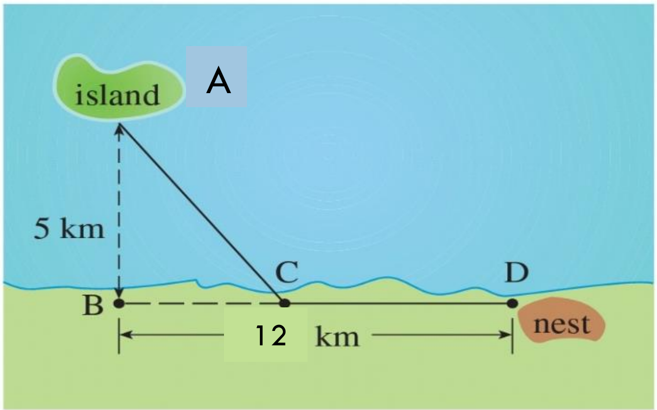
   c) f ' changes from positive to negative   d) f '' < 0

5. The shortest distance is

   a) 10 km   b) $\sqrt{125}\ km$ $\sqrt{133}\ km$ $\sqrt{160}\ km$

VII. Ornithologists(bird watchers) have found that some species of birds tend to avoid flights over large bodies of water during day time. It is believed that more energy is required to fly over water than over land because due to heat air generally rises over land and falls over water during the day.

A bird with these tendencies is released from an island (A) 5 km from the nearest point B on a straight shoreline, flies to a point C on the shoreline and then flies along the shoreline to its nesting area D.



Assume that the bird instinctively chooses a path that will minimize its energy expenditure. Points B and D are 12 km apart.

It is given that the energy required for the bird to fly over water is 1.5 times the energy required to fly over land.

Assume that, the energy/km over land = k.

Then, energy/km over water = 1.5k. Let, BC = x.

Now, answer the following questions :

1. The function F for the energy expenditure which is to be minimized is

a) F(x) = $\sqrt{x^2+25} + (12-x)$

b) F(x) = 1.5k$\sqrt{x^2+25} + k(12-x)$

c) F(x) =1.5k[$\sqrt{x^2+25} + (12-x)$]

d) F(x) = k[$\sqrt{x^2+25} + (12-x)$]

2. The critical point is

a) x = 5 b) x = $\sqrt{5}$ c) x = 2 $\sqrt{5}$ d) x = $2\sqrt{10}$

3. The total distance the bird has to fly is(in km)

a) $\sqrt{45}$ b) 12 - $\sqrt{5}$ c) 12 + $\sqrt{5}$ d) 12+ $\sqrt{10}$

4. The total energy requirement is(approximately)

a) 14.8k b) 15k c) 17.6k d) 16.2k

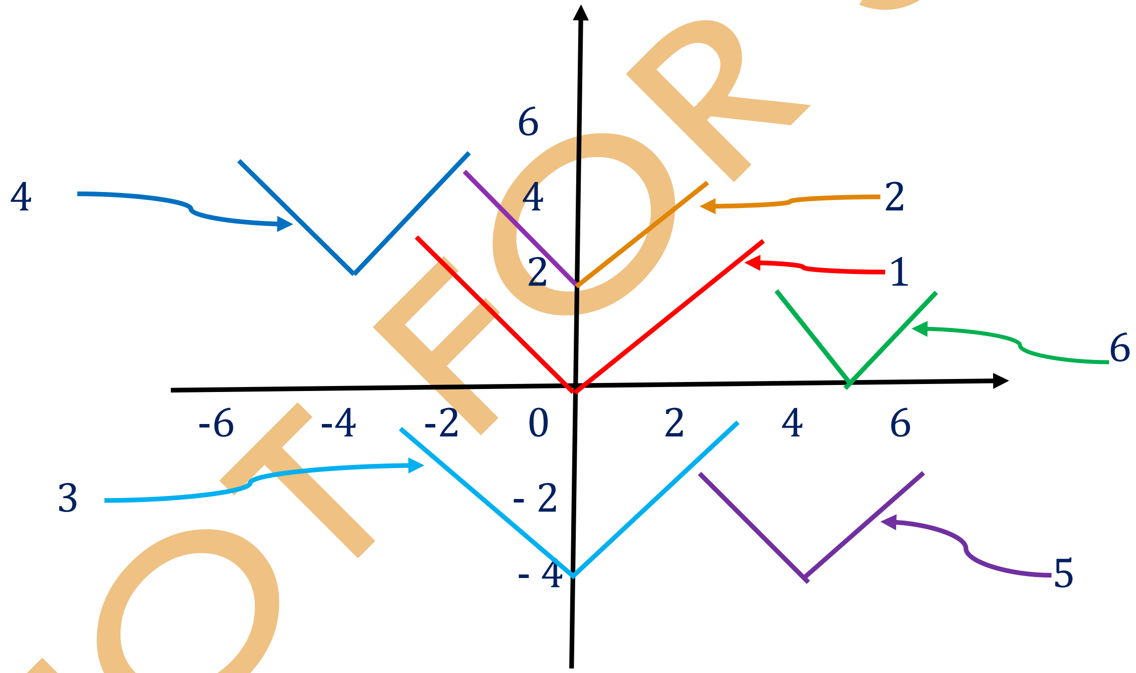
5. If the bird were to fly along AD, then how much extra energy would be required

a) 1.9k b) 5k c) 3.3k d) 4.7k

VI. APPLICATIONS OF DIFFERENTIATION-SOLUTIONS

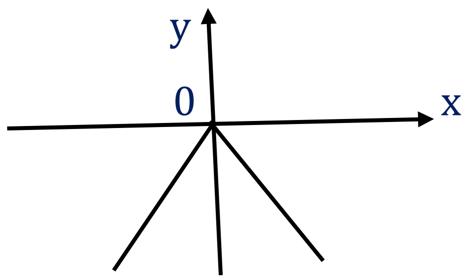
THE FOLLOWING GRAPHS ARE TO MAKE YOUR THINKING EASIER IN SOLVING THE PROBLEMS OF THE TYPE WITHOUT RESORTING TO ALGEBRAIC METHODS .

TRANSFORMATIONS OF THE GRAPH OF Y = |X|



1. y = |x|
2. y – 2 = |x| or y = |x| + 2
3. y + 4 = |x| or y = |x| - 4
4. y – 2 = |x + 4| or y = |x + 4| + 2
5. y + 4 = |x - 4| or y = |x - 4| - 4
6. y = |x -5|

GRAPH OF Y = - |X|



The figure on the right shows the graph of y = -|x| which is obtained by taking the reflection of the graph of y = |x| about the x-axis.

Use the transformation techniques for any type of function.

NOTE :

1. Remember that f(x) = |x| has minimum value 0 and no maximum.
2. And that f(x) = - |x| has maximum value 0 and no minimum.
3. The graph of y = f(x) = |x ± k| can be obtained from y = f(x) = |x| by moving to the right(when x – k) and moving to the left (when x +k).
4. Similarly y - k makes shift upwards and y + k makes shift downwards, by k units.
5. As is clear, horizontal shift does not induce any change in maximum or minimum whereas a vertical shift effects a change.

VI. SOLUTIONS

1. The maximum value of f(x) = $4 - |x|$.

   The equation is y = 4 - |x| or y – 4 = - |x|.

   The maximum value of y = - |x| is 0.

   y – 4 = - |x| means the graph is moved up along positive y-axis by 4 units. ∴ The maximum value of f(x) is 4.

2. The minimum value of $f(x) = |2x + 3|$.

The graph of f(x) is similar to y = |x| but has its corner point at x = - 3/2. ∴ The minimum value of f(x) is -3/2.

3. The minimum value of $f(x) = |2x - 1|$.

The minimum value of f(x) is 0.

4. The minimum value of $f(x) = 2|x|$. The minimum value is 0.

5. absolute maximum value of $f(x) = 3x^2$, $[-1, 2]$.

$f'(x) = 0 \Rightarrow 6x = 0$; $x = 0$.The critical number is 0. The end points are – 1 and 2. Now, f(0) = 0, f(- 1) = 3, f(2) = 12.

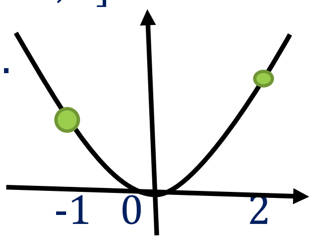
Therefore, the absolute maximum value is 12.

6. Find the absolute minimum value of $f(x) = 3x^2$, $[-1, 2]$.

$f'(x) = 0 \Rightarrow 6x = 0$; $x = 0$.The critical number is 0.

The end points are – 1 and 2.

Now, f(0) = 0, f(- 1) = 3, f(2) = 12.



Therefore, the absolute minimum value is 0. Graph for Qns. 5 & 6.

7. The absolute maximum value of $f(x) = 2x - 3$ on [0,2].

$f'(x) = 2$, which is a constant. No critical point.

The end points are 0, 2. f(0) = - 3, f(2) = 1

Therefore, the absolute minimum value is 1.

8. $f(x) = \begin{cases} 2x + 2, & 0 \le x < 1 \\ 4x^2, & 1 \le x \le 3 \end{cases}$.

9. the maximum value of $f(x) = |\sin x|$, $x \in [0, 2\pi]$

10. the maximum value of $f(x) = \sin x$, $x \in [\pi, 2\pi]$

$f'(x) = \cos x = 0 \Rightarrow x = \frac{3\pi}{2}$; The critical point is $\frac{3\pi}{2}$.

The end points are $\pi, 2\pi$. $F(\pi) = 0$; $f(2\pi) = 0$; $f(\frac{3\pi}{2}) = -1$.

The absolute maximum value is 0.

11. The maximum value of f(x) = $-|x+2|+3$; y – 3 = - |x +2|

For y = -|x| and y = -|x+2|, the maximum value is 0.

The maximum value of f(x) is 3.

12. The maximum value of f(x) = $-|x-4|-5$; y + 5 = -|x – 4|

The maximum value of f(x) is – 5.

f(x) = - |x|	standard position	max. 0
f(x) = - |x – 4|	right shift 4 units	max. 0
f(x) + 5 = - |x – 4|	downward shift 5 units	max. – 5.

13. the maximum value of f(x) = $1-x^2$. f ’(x) = - 2x. f ’(x) = 0 ; x = 0.

f ”(0) = - 2 < 0. So, f is max. at x = 0. f(0) = 1 is the maximum value.

14. the minimum value of f(x) = x^2-8. Proceeding as in Qn. 13,

f if minimum at x = 0. The minimum value is – 8.

15. the maximum value of f(x) = sin x – 3.

f ’(x) = cos x = 0 ⇒ x = $(2n+1)\frac{\pi}{2}$; $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}...$

The maximum occurs at infinitely many points, as sin is periodic.

The max. value is f($\frac{\pi}{2}$) $= 1 - 3 = -2$.

Alternatively, Using the fact that sin x lies between – 1 and 1,

$-1 \le \sin x \le 1$; $-1-3 \le \sin x - 3 \le 1-3$;

$-4 \le \sin x - 3 \le -2$. Therefore, max. value of f(x) is – 2.

16. the minimum value of f(x) = sin x +2 . Proceeding as in Qn. 15,

the minimum value of f(x) = 1.

17. f(x) = cos x + 4. Use the fact that cos x lies between – 1 and 1.

The maximum value of f(x) is 5.

18. f(x) = cos x – 5. The minimum value of f(x) is – 6.

19. f(x) = 2sin x – 3. Use, $-1 \le \sin x \le 1$; $-2 \le 2\sin x \le 2$;

$-5 \leq 2\sin x - 3 \leq -1$; So, the maximum value of f(x) is – 1.

20. f(x) = 4sin x +1 ; Proceeding as in Qn. 19, min. of f(x) is – 3.

21. $f(x) = e^x$; $f'(x) = e^x > 0$, for all $x \in R$. [$e \cong 2.718$]

$\therefore$ e^x is increasing everywhere(or always or throughout or for all values of x). In fact, e^x is strictly increasing.

22. $f(x) = e^{-x}$. $F'(x) = -e^{-x} < 0$, for all $x \in R$.

$\therefore$ e^{-x} is decreasing everywhere.(also, strictly).

23. f(x) = log x. The domain of f(x) is $(0, \infty)$. $f'(x) = \frac{1}{x} > 0, \forall x \in (0, \infty)$.

$\therefore$ log x is increasing in $(0, \infty)$. (also, strictly).

24. $f(x) = x + 2$

$f'(x) = 1 > 0$. So, f(x) is strictly increasing on R.

25. $f(x) = 6 - 2x$

$f'(x) = -2 < 0$, for all real x. So, f(x) is strictly decreasing on R.

26. $f(x) = x^2 + 5$

$f'(x) = 2x = 0 \Rightarrow x = 0$. Consider the sub intervals $(-\infty, 0)$, $(0, \infty)$.

In $(-\infty, 0)$, $f' < 0$,

In $(0, \infty)$, $f' > 0$. $\therefore$ f(x) is strictly increasing in $(0, \infty)$.

27. $f(x) = -x^2$

$f'(x) = -2x = 0 \Rightarrow x = 0$. Consider the sub intervals $(-\infty, 0)$, $(0, \infty)$.

In $(-\infty, 0)$, $f' > 0$,

In $(0, \infty)$, $f' < 0$. $\therefore$ f(x) is strictly increasing in $(-\infty, 0)$.

28. $f(x) = x^2$

$f'(x) = 2x = 0 \Rightarrow x = 0$. Consider the sub intervals $(-\infty, 0)$, $(0, \infty)$.

In $(-\infty, 0)$, $f' < 0$,

In $(0, \infty)$, $f' > 0$. $\therefore$ f(x) is strictly decreasing in $(-\infty, 0)$.

29. $f(x) = -x^2 + 1$

f '(x) = - 2x = 0 ⇒ x = 0. Consider the sub intervals $(-\infty, 0)$, $(0, \infty)$.

In $(-\infty, 0)$, f ' > 0 ,

In $(0, \infty)$, f ' < 0. ∴ f(x) is strictly decreasing in $(0 , \infty)$.

30. the interval in which f(x) = e^{-x^2} is strictly increasing. ?

f ' (x) = $e^{-x^2}.(-2x)$; f ' (x) = 0 ⇒ - 2x = 0 ($e^{-x^2} \neq 0$)

⇒ x = 0. In $(-\infty, 0)$, f ' > 0 and in $(0, \infty)$, f ' < 0.

∴ f(x) is strictly increasing in $(-\infty, 0)$.[NOTE:$e^{-x^2} > 0, \forall\ real\ x$]

31. If f(x) = cos x is strictly decreasing on (0 , a) , find the value of "a".

f ' (x) = - sin x ; x = 0, $n\pi$. $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi \ldots$

There are many sub-intervals in which f(x) is st.dec.

In $(0, \frac{\pi}{2})$, cos x is st.dec. So, a = $\frac{\pi}{2}$.

32. f(x) = $2x^2 + 2x$

f '(x) = 4x +2 = 0 ⇒ $x = -\frac{1}{2}$. The intervals are $\left(-\infty, -\frac{1}{2}\right)$, $\left(-\frac{1}{2}, \infty\right)$.

In the interval $\left(-\infty, -\frac{1}{2}\right)$, f ' < 0 and in $\left(-\frac{1}{2}, \infty\right)$, f ' > 0.

∴ f(x) is strictly increasing in $\left(-\frac{1}{2}, \infty\right)$.

33. f(x) = tan x , $[0, \pi]$. In the interval $[0, \pi]$, f(x) is discontinuous at x = $\frac{\pi}{2}$. A function should be continuous in order to have an absolute maximum or absolute minimum on a closed interval. So, no absolute maximum and absolute minimum.

34. y^2 = 4ax. Differentiating, 2yy ' = 4a ;

$y' = \frac{dy}{dx} = \frac{2a}{y}$ *slope of tangent*.

Slope of normal = $\frac{-y}{2a}$. ∴ At the origin, slope of normal = 0.

35. The equation of tangent to the curve y = sin x at the origin.

y ‘ = cos x. Slope of tangent = y’ = $\frac{dy}{dx}$ at (0,0) = cos 0 = 1.

∴ The eqn. of tangent is y – 0 = 1(x – 0). That is., y = x is the tangent.

36. the equation of normal to the curve y = sin x at the origin.

y ‘ = cos x. Slope of tangent = y’ = $\frac{dy}{dx}$ at (0,0) = cos 0 = 1.

Slope of normal = - 1. ∴ The eqn. of normal is y – 0 = - 1(x – 0).

That is., y = - x is the normal.

37. the equation of tangent to the curve y = cos x at $\left(\frac{3\pi}{2}, 0\right)$.

y ‘ = - sin x ; Slope of tangent = y’ = $\frac{dy}{dx}$ at $\left(\frac{3\pi}{2}, 0\right) = -sin\,\frac{3\pi}{2}$ = 1.

∴ The eqn. of tangent is y – 0 = 1(x – $\frac{3\pi}{2}$) ; x – y – $\frac{3\pi}{2}$ = 0.

38. y = e^x ; y ‘ = e^x . At the point where the curve cuts y-axis,

x = 0 ; y = 1. The point is (0,1).

Slope of tangent = $\frac{dy}{dx}$ at (0,1) = e^0 = 1 = tan θ $\Rightarrow$ $\theta = \frac{\pi}{4}$.

39. y = e^x ; y ‘ = e^x . At the point where curve cuts y-axis, x = 0 ; y = 1.

The point is (0,1). Slope of tangent = $\frac{dy}{dx}$ at (0,1) = e^0 = 1.

40. y = e^x ; y ‘ = e^x . At the point where the curve cuts y-axis,

x = 0 ; y = 1. The point is (0,1).

Slope of tangent = $\frac{dy}{dx}$ at (0,1) = e^0 = 1. Slope of normal = - 1.

41. y = e^x ; y ‘ = e^x . At the point where curve cuts y-axis, x = 0 ; y = 1.

The point is (0,1). Slope of tangent = $\frac{dy}{dx}$ at (0,1) = e^0 = 1.

∴ The eqn. of tangent is y – 1 = 1(x – 0). i.e., x – y + 1 = 0.

42. y = e^x ; y ‘ = e^x . At the point where the curve cuts y-axis,
x = 0 ; y = 1. The point is (0,1).
Slope of tangent = $\frac{dy}{dx}$ at (0,1) = e^0 = 1. Slope of normal = - 1.

∴ The eqn. of normal is y – 1 = - 1(x – 0). i.e., x + y - 1 = 0.

43. y = e^{-x} ; y ' = $-e^{-x}$. At the point where the curve cuts y-axis, x = 0 ; y = 1. The point is (0,1).

Slope of tangent = $\frac{dy}{dx}$ at (0,1) = e^0 = - 1.

44. y = log x ; y ' = $\frac{1}{x}$. At the point where curve cuts x-axis, y = 0 ; x = 1.

The point is (1,0). Slope of tangent = $\frac{dy}{dx}$ at (1,0) =1/1 = 1.

45. y = $\frac{1}{x-3}$; Differentiating, $\frac{dy}{dx} = \frac{-1}{(x-3)^2}$; slope = - 1.

∴ $\frac{-1}{(x-3)^2} = -1 \Rightarrow (x-3)^2 = 1 \Rightarrow x^2 - 6x + 8 = 0,$

which gives x = 2,4. So, two tangents can be drawn.

46. 4y ' = 2x ⇒ y ' = $\frac{dy}{dx} = \frac{x}{2}$; $Slope\ of\ tangent = \left(\frac{dy}{dx}\right)_{(0,0)} = 0.$

So, slope of normal = ∞

47. 4y ' = 2x ⇒ y ' = $\frac{dy}{dx} = \frac{x}{2}$; $Slope\ of\ tangent = \left(\frac{dy}{dx}\right)_{(0,0)} = 0.$

So, slope of normal = ∞. ∴ The eqn. of normal is y – 0 = $\frac{1}{0}$(x – 0)

That is., x = 0 is the equation of normal.

48. 4y ' = 2x ⇒ y ' = $\frac{dy}{dx} = \frac{x}{2}$; $Slope\ of\ tangent = \left(\frac{dy}{dx}\right)_{(0,0)} = 0.$

∴ The eqn. of tangent is y – 0 = 0 (x – 0) ; y = 0.

49. y = cos x ; y ' = - sin x ; $Slope\ of\ tangent = \left(\frac{dy}{dx}\right)_{(x = \pi/2)} = -1.$

So, slope of normal = 1.

50. $\frac{dy}{dx} = be^{-x/a} . \left(\frac{-1}{a}\right)$; When x = 0, $\frac{dy}{dx} = be^{-0/a} . \left(\frac{-1}{a}\right) = \frac{-b}{a}$

So, slope of the tangent = $\frac{-b}{a}$.

51. Differentiating, $\frac{dx}{dt} = 2at$; $\frac{dy}{dt} = 2a$; $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}$

When t = 1 , $Slope\ of\ tangent = \left(\frac{dy}{dx}\right)_{(t = 1)} = 1.$

52. The absolute maximum and minimum of $f(x) = 24, x \in [4,12]$.

$f(x)$ is a constant function. No critical points.

24 is the absolute maximum as well as the absolute minimum.

VI. APPLICATIONS OF DIFFERENTIATION

SOLUTIONS TO CASE STUDIES

I.	**1. B**	**2. c**	**3. c**	**4. d**	**5. d**
II.	**1. b**	**2. b**	**3. b**	**4. b**	**5. a**
III.	**1. a**	**2. c**	**3. a**	**4. b**	**5. c**
IV.	**1. b**	**2. c**	**3. d**	**4. c**	**5. b**
V.	**1. b**	**2. b**	**3. a**	**4. a**	**5. b**
VI.	**1. b**	**2. c**	**3. d**	**4. a**	**5. b**
VII.	**1. b**	**2. c**	**3. c**	**4. c**	**5. a**

END OF CHAPTER 6

// K.MANI , SALEM, TAMILNADU //

VII. INTEGRALS

VERY SHORT ANSWER TYPE

EVALUATE THE FOLLOWING INTEGRALS :

1. $\int \pi^x \ dx.$
2. $\int a^{\log x} \ dx.$
3. $\int \frac{x-1}{x+1} dx$.
4. $\int e^x \sqrt{1 + e^x} \ dx$.
5. $\int \tan^3 x \ \ dx$.
6. $\int \frac{X}{\sqrt{X+1}} \ dx.$
7. $\int \frac{1}{x\sqrt{4x^2 - 1}} dx$.
8. $\int x\sqrt{x + 1} \ dx$.
9. $\int \frac{e^x}{e^x + 1} dx$.
10. $\int \sin^{-1}(\cos \alpha x) \ dx$.
11. $\int \sin^{-1}(\cos \alpha) \ dx$.
12. $\int \frac{1}{9 - 4x^2} \ dx$.
13. $\int \frac{x}{x + 1} dx$.
14. $\int \frac{x^2 - 1}{x^2 + 1} dx$.
15. $\int \frac{1 + \log x}{x} dx.$
16. $\int x \sin (e^{x^2}) \, e^{x^2} \ dx$.

17. $\int \sqrt{x}\, cos(x\sqrt{x})\, dx$.

18. $\int \frac{\sqrt{1 + \sqrt{x}}}{\sqrt{x}}\, dx$.

19. $\int \frac{1}{1 + \sqrt{x}}\, dx$.

20. $\int \frac{sin^4 x + cos^4 x}{sin^2 x + cos^2 x}\, dx$.

21. $\int \frac{sin^3 x + cos^3 x}{sin^2 x\ cos^2 x}\, dx$.

22. $\int \frac{cos\, 2x}{cos\, x}\, dx$.

23. $\int \frac{\sqrt{sin^3 x}}{sec\, x}\, dx$.

24. $\int \frac{x - 1}{(x + 1)^2}\, dx$.

25. $\int \frac{tan\, x}{log\, sec\, x}\, dx$.

26. $\int cos\, x\, sin(sin\, x)\, dx$.

27. $\int \frac{1}{x^2\, e^{5/x}}\, dx$.

28. $\int \frac{2^x}{e^x}\, dx$.

29. $\int \frac{sin\,(x + \alpha)}{sin\, x}\, dx$.

30. $\int e^x\,(sin\, x - cos\, x)\, dx$.

31. $\int e^x \left(\frac{1 + 2x}{\sqrt{x}}\right) dx$.

32. $\int e^{(x + \sqrt{x})} \left(1 + \frac{1}{2\sqrt{x}}\right) dx$.

33. $\int e^x\ (log\, x + x log\, x + 1)\, dx$.

34. $\int \frac{sin\, x}{\sqrt{cos\, x}}\, dx$.

35. $\int \frac{\sqrt{4x^5 - x^4}}{x^2}\ dx$.

36. $\int \frac{2x + 3}{(x + 7)^3}\ dx$.

37. $\int \frac{1}{\sqrt{1 + \sqrt{x}}}\ dx$.

38. $\int \frac{1}{(1 + \sqrt{x})^3}\ dx$.

39. $\int \frac{\log x^2}{x}\ dx$.

40. $\int \frac{1}{x (\log x)^n}\ dx$.

41. $\int \frac{x^2 + 2x}{(x + 1)^2}\ dx$.

42. $\int_0^{3/2} |\sin \pi x|\ dx$.

43. $\int_0^1 |\cos \pi x|\ dx$.

44. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin x|\ dx$.

45. $\int_0^{\pi} |\cos x|\ dx$.

46. $\int_{-2}^{0} [x]\ dx$.

47. $\int_0^{\frac{3}{2}} [x]\ dx$.

48. $\int_{-1}^{1} [x]$.

49. $\int_{-1}^{0} \frac{1}{|x|}\ dx$.

50. $\int_{-1}^{1} x^3\ dx$.

51. $\int_{-\frac{\pi}{2}}^{\pi} \sin x\ dx$.

52. $\int_{-\pi}^{\pi} \cos x\ dx$.

53. $\int_{-1}^{2} x|x|\, dx$.

54. $\int_{-1}^{1} |x^2|\, dx$.

55. $\int_{-1}^{1} |x^3|\, dx$.

56. $\int_{-3}^{-2} |x^3|\, dx$.

57. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} tan^2 x\, dx$.

58. $\int_{-3}^{-2} \frac{1}{x}\, dx$.

59. $\int_{-2}^{2} |1 - x|\, dx$.

60. $\int_{-1}^{1} \left[\frac{x}{2}\right] dx$.

61. $\int_{-1}^{1} (x - [x])\, dx$.

62. $\int_{1}^{2} [x^2]\, dx$.

63. $\int_{-1}^{1} x|x|\, dx$.

64. $\int_{-1}^{1} x[x]\, dx$.

65. $\int_{0}^{2\pi} sin\, x\, dx$.

66. $\int_{0}^{2\pi} cos\, x\, dx$.

67. $\int_{-\pi/4}^{\pi/4} tan\, x\, dx$.

68. $\int_{-1}^{1} |x^2 - 1|\, dx$.

69. $\int_{-2}^{0} |x^2 - 1|\, dx$.

70. $\int_{1}^{2} |x^2 - 1|\, dx$.

71. $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$.

72. $\int_0^{\pi} \sin |x| \, dx$.

73. $\int_0^{\pi} \cos |x| \, dx$.

74. $\int_{-1}^{1} (1 - |x|) \, dx$.

75. $\int_{-a}^{a} (a - |x|) \, dx$.

76. $\int_0^2 |2x - 1| \, dx$.

77. $\int_1^e \frac{1}{x} \, dx$.

78. $\int_0^1 \frac{x^3 + 8}{x + 2} \, dx$.

79. $\int_0^3 |1 - x^2| \, dx$.

80. $\int_0^2 |x^3 - 1| \, dx$.

81. $\int_{-1}^{1} |x - x^2| \, dx$.

82. $\int_{-4}^{5} f(x) \, dx$, where $f(x) = \begin{cases} 9, & x < 3 \\ x^2, & x \geq 3 \end{cases}$

83. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin^3 x \cos x + \cos x \sin x) dx$

VII. INTEGRALS – SOLUTIONS

1. $\int \pi^x \ dx. = \frac{\pi^x}{\log \pi} + c.$

2. $\int a^{\log x} \ dx \ = \int a^t e^t \ dt$

$= \int (ae)^{\ t} \ dt$

$x \, dt = dx$

$= \frac{(ae)^{\log x}}{1 + \log a} + c \, .$

Put, $t = \log x$

$dt = \frac{1}{x} \ d \quad = \frac{(ae)^t}{\log ae} + c$

$e^t \ dt \ = \ dx$

3. $\int \frac{x-1}{x+1} \, dx = \int \frac{x \ + 1 - 2}{x+1} \, dx = \int \frac{x \ + 1 - 2}{x+1} \, dx \ = \int 1 - \frac{2}{x+1} \, dx$

$= x - 2\log(x + 1) + c.$

4. $\int e^x \ \sqrt{1 \ + \ e^x} \ \ dx = \int \sqrt{t} \ \ dt = \frac{2}{3} t^{3/2} \ + \ c.$

$= \ \frac{2}{3}(1 \ + \ e^x)^{3/2} \ + \ c.$

put $t = 1 + e^x$

$dt = e^x \, dx$

5. $\int \tan^3 x \ \ dx = \int \tan^2 x \tan x \ \ dx = \int (\sec^2 x \ - \ 1) \tan x \ \ dx$

$= \int (\sec^2 x \ \tan x \ \ dx - \int \tan x \, dx = \frac{(\tan x)^2}{2} - \log|\sec x| + \ c$

6. $\int \frac{X}{\sqrt{X+1}} \, dx = \int \frac{X + 1 - 1}{\sqrt{X+1}} \, dx. \ = \ \int \sqrt{x \ + \ 1} \ - \ \frac{1}{\sqrt{x+1}} \ dx$

$= \int (x+1)^{1/2} \ - \ (x+1)^{-1/2} \, dx.$

$= \frac{2}{3}(x+1)^{3/2} \ - \ 2(x+1)^{1/2} \ + \ c.$

7. $\int \frac{1}{x\sqrt{4x^2 \ - \ 1}} \, dx \, . = \ 2\int \frac{1}{2x\sqrt{(2x)^2 \ - \ 1}} \, dx \, . = 2.\frac{1}{2} \sec^{-1}(2x) \ + \ c = \sec^{-1}(2x)$

8. $\int x\sqrt{x \ + \ 1} \ dx \, . = \int (t \ - \ 1) \sqrt{t} \, dt \ \ = \ \int t^{3/2} \ - \ t^{1/2} \, dt$

$= \frac{2}{5} t^{5/2} \ - \frac{2}{3} t^{3/2} \ + \ c = \frac{2}{5}(1 + x)^{5/2} \ - \frac{2}{3}(1 + x)^{3/2} \ + \ c.$

put $t = x + 1$

$dt = dx.$

9. $\int \frac{e^x}{e^x \ + 1} \ dx \, . = \int \frac{dt}{t} = \log|t| \ = \ \log|1 \ + \ e^x| \ + \ c.$ [put $t = 1 + e^x$; $dt = e^x \, dx$.]

10. $\int \sin^{-1}(\cos \alpha x)\, dx = \int \sin^{-1} \sin(\frac{\pi}{2} - \alpha x)\, dx = \int (\frac{\pi}{2} - \alpha x) dx = \frac{\pi}{2}x - \frac{\alpha x^2}{2} + c.$

11. $\int \sin^{-1}(\cos \alpha)\, dx = \int \sin^{-1} \sin(\frac{\pi}{2} - \alpha)\, dx = \int (\frac{\pi}{2} - \alpha) dx = \frac{\pi}{2}x - \alpha x + c\,.$

12. $\int \frac{1}{9 - 4x^2} dx = \frac{1}{4} \int \frac{1}{\left(\frac{3}{2}\right)^2 - x^2}\, dx = \frac{1}{4} \cdot \frac{1}{2 \cdot \frac{3}{2}} \log \frac{\frac{3}{2} + x}{\frac{3}{2} - x} + c = \frac{1}{12} \log \frac{3 + 2x}{3 - 2x} + c.$

13. $\int \frac{x}{1 + x}\, dx . = \int \frac{1 + x - 1}{x + 1}\, dx = \int 1 - \frac{1}{1 + x}\, dx = x - \log(1 + x) + c$

14. $\int \frac{x^2 - 1}{x^2 + 1}\, dx . = \int \frac{x^2 + 1 - 2}{x^2 + 1}\, dx . = \int 1 - \frac{2}{x^2 + 1}\, dx = x - 2 \tan^{-1} x + c.$

15. $\int \frac{1 + \log x}{x}\, dx = \int (1 + \log x) \cdot \frac{1}{x}\, dx =$ | put $t = 1 + \log x$

$\int t\, dt = \frac{t^2}{2} = \frac{(1 + \log x)^2}{2} + c.$ | $dt = \frac{1}{x}\, dx$

16. $\int x \sin(e^{x^2})\, e^{x^2}\, dx = \int \sin t\, \frac{dt}{2} = -\frac{1}{2}\cos(e^{x^2}) + c.$ | [Put, $t = e^{x^2}$;

$dt = 2xe^{x^2} dx$]

17. $\int \sqrt{x} \cos(x\sqrt{x})\, dx = \int x^{1/2} \cos\left(x^{3/2}\right) dx =$ | Put $t = x^{3/2}$

$= \int \cos t \cdot \frac{2}{3}\, dt = \frac{2}{3} \cdot \sin t = \frac{2}{3} \cdot \sin\left(x^{3/2}\right) + c.$ | $dt = \frac{3}{2} x^{1/2} dx\, ; \frac{2}{3} dt = x^{1/2} dx$

18. $\int \frac{\sqrt{1 + \sqrt{x}}}{\sqrt{x}}\, dx = \int \sqrt{1 + \sqrt{x}} \cdot \frac{1}{\sqrt{x}}\, dx$ | Put $t = 1 + \sqrt{x}$;

$= \int \sqrt{t}\ 2dt = 2 \cdot \frac{t^{3/2}}{3/2}$ | $dt = \frac{1}{2\sqrt{x}} dx\, ; 2\, dt = \frac{1}{\sqrt{x}}\, dx$

$= \frac{4}{3} \cdot \left(1 + \sqrt{x}\right)^{3/2} + c.$

19. $\int \frac{1}{1 + \sqrt{x}}\, dx . = \int \frac{\sqrt{x}}{1 + \sqrt{x}} \cdot \frac{1}{\sqrt{x}}\, dx = \int \frac{t - 1}{t}\, 2\, dt$ | Put, $t = 1 + \sqrt{x}$;

$= 2 \int 1 - \frac{1}{t}\, dt = 2(t - \log t)$ | $dt = \frac{1}{2\sqrt{x}}\, dx$

$= 2[(1 + \sqrt{x}) - \log(1 + \sqrt{x})] + c$ | $2\, dt = \frac{1}{\sqrt{x}}\, dx$

20. $\int \frac{\sin^4 x + \cos^4 x}{\sin^2 x \cos^2 x} dx. = \int \frac{s^4}{s^2c^2} + \frac{c^4}{s^2c^2} dx = \int tan^2x + cot^2x \ dx$

$= \int sec^2x - 1 + cosec^2x - 1 \, dx = tanx - cot\,x - 2x + c.$

21. $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x} dx = \int \frac{s^3}{s^2c^2} + \frac{c^3}{s^2c^2} dx =$

$\int secx\,tanx + cotx\,cosecx \ dx = \sec x - \operatorname{cosec} x + c.$

22. $\int \frac{\cos 2x}{\cos x} dx = \int \frac{2cos^2x - 1}{cos\,x} dx = \int 2cosx - secx \ dx$

$= 2\sin x - \log|\sec x + \tan x| + c.$

23. $\int \frac{\sqrt{\sin^3 x}}{\sec x} dx = \int (sin\,x)^{3/2} \ cos\,x\,dx = \int t^{3/2}\,dt$ | Put $t = \sin x$

$= \frac{2}{5} \cdot t^{5/2} = \frac{2}{5} \cdot (sinx)^{5/2} + c.$ | $dt = \cos x \, dx$

24. $\int \frac{x - 1}{(x + 1)^2} dx = \int \frac{x + 1 - 2}{(x + 1)^2} dx = \int \frac{1}{x + 1} - 2(x + 1)^{-2} dx$

$= \log|x + 1| - 2\frac{(x + 1)^{-1}}{-1} = \log|x + 1| + \frac{2}{x + 1} + C.$

25. $\int \frac{tan\,x}{log\,sec\,x} dx. = \int \frac{dt}{t} = log\,|t| = log|\,log\,sec\,x| + c$

26. $\int cos\,x\,sin(sin\,x)\,dx = \int sint\,dt = -cos\,t = -cos(sin\,x) + c.$ [t = sinx]

27. $\int \frac{1}{x^2\,e^{5/x}} dx = \int \frac{1}{x^2} e^{-5/x}\,dx = \int e^t \frac{dt}{5}$ | Put, $t = -\frac{5}{x}$; $dt = \frac{5}{x^2}dx$

$= \frac{1}{5}e^t = \frac{1}{5}e^{-5/x} + c.$ | $\frac{dt}{5} = \frac{1}{x^2}dx$

28. $\int \frac{2^x}{e^x} dx = \int \left(\frac{2}{e}\right)^x dx = \frac{\left(\frac{2}{e}\right)^x}{\log\left(\frac{2}{e}\right)} = \frac{2^x e^{-x}}{log2 - loge} + c.$

29. $\int \frac{\sin(x+\alpha)}{\sin x}\,dx = \int \frac{\sin x \cos\alpha + \cos x \sin\alpha}{\sin x}\,dx = \int \cos\alpha + \sin\alpha \cot x \; dx$

$= x\cos\alpha + \sin\alpha \log \sin x + c.$

30. $\int e^x(\sin x - \cos x)dx = -\int e^x[\cos x + (-\sin x)]dx = -e^x \cos x + c.$

31. $\int e^x\left(\frac{1+2x}{\sqrt{x}}\right) dx = \int e^x\left(\frac{1}{\sqrt{x}} + 2\sqrt{x}\right) dx$

$= e^x \cdot 2\sqrt{x} + c.$ [here, $f = 2\sqrt{x}$]

32. $\int e^{(x+\sqrt{x})}\left(1 + \frac{1}{2\sqrt{x}}\right) dx = \int e^t\,dt = e^t = e^{x+\sqrt{x}} + c.$

[Put, $t = x + \sqrt{x}$].

33. $\int e^x(\log x + x\log x + 1)dx$

$= \int e^x\{(x\log x) + (1+\log x)\}\,dx$ [put $f = x\log x$]

$= \int e^x(f + f')\,dx = e^x f$

$= e^x\, x\log x + c.$

34. $\int \frac{\sin x}{\sqrt{\cos x}}\,dx = \int \frac{-dt}{\sqrt{t}} = -2\int \frac{1}{2\sqrt{t}}\,dt$ ‖ put, $t = \cos x$

$= -2\sqrt{t} = -2\sqrt{\cos x} + c.$ ‖ $dt = -\sin x\,dx$

35. $\int \frac{\sqrt{4x^5 - x^4}}{x^2}\,dx. = \int \sqrt{\frac{4x^5 - x^4}{x^4}}\,dx = \int \sqrt{4x-1}\,dx$

$= \frac{1}{4}\frac{(4x-1)^{3/2}}{3/2} = \frac{1}{6}(4x-1)^{3/2} + c.$

36. $\int \frac{2x+3}{(x+7)^3}\,dx = \int \frac{2(t-7)+3}{t^3} = \int \frac{2t-11}{t^3}\,dt$ ‖ put $t = x+7$,

$x = t - 7$

$= \int \frac{2}{t^2} - \frac{11}{t^3}\,dt = \frac{-2}{t} + \frac{11}{2t^2} = \frac{-2}{x+7} + \frac{11}{2(x+7)^2} + c.$ ‖ $dt = dx$

37. $\int \frac{1}{\sqrt{1+\sqrt{x}}}\,dx = \int \frac{1}{\sqrt{t}}\, 2\sqrt{x}\; dt = 2\int \frac{t-1}{\sqrt{t}}\, dt$ | put $t = 1+\sqrt{x}$

$= \int \sqrt{t} - \frac{1}{\sqrt{t}}\, dt = 2\left[\frac{t^{3/2}}{3/2} - 2\sqrt{t}\right]$ | $dt = \frac{1}{2\sqrt{x}}\, 2\, ;\, 2\sqrt{x}\; dt = dx$

$= 2\left\{\frac{2}{3}(1+\sqrt{x})^{3/2} - 2\sqrt{1+\sqrt{x}}\right\} + c$

38. $\int \frac{1}{(1+\sqrt{x})^3}\,dx = \int \frac{2(t-1)}{t^3}\,dt = 2\int \frac{1}{t^2} - \frac{1}{t^3}\,dt$ | put $t = 1+\sqrt{x}$; $dt = \frac{1}{2\sqrt{x}}\,dx$

$= 2\left\{\frac{-1}{t} + \frac{1}{2t^2}\right\} = 2\left\{\frac{-1}{1+\sqrt{x}} + \frac{1}{2(1+\sqrt{x})^2}\right\} + c$ | $2\sqrt{x}\; dt = dx; 2(t-1)dt = dx$

39. $\int \frac{\log x^2}{x}\,dx = 2\int \frac{\log x}{x}\,dx = 2\int t\,dt$ | put, $t = \log x$; $dt = \frac{1}{x}\,dx$

$= 2\frac{t^2}{2} = (\log x)^2 + c$

40. $\int \frac{1}{x(\log x)^n}\,dx = \int \frac{1}{t^n}\,dt = \frac{t^{-n+1}}{-n+1} = \frac{(\log x)^{1-n}}{1-n} + c$. [put $t = \log x$; $dt = \frac{1}{x}\,dx$]

41. $\int \frac{x^2+2x}{(x+1)^2}\,dx = \int \frac{x^2+2x+1-1}{(x+1)^2}\,dx = \int 1 - \frac{1}{(x+1)^2}\,dx$

$= \int 1 - (x+1)^{-2}\,dx = x + \frac{1}{x+1} + c.$

42. $\int_0^{3/2} |\sin \pi x|\,dx = I$, say. $|\sin \pi x| = \begin{cases} \sin \pi x, 0 < x < 1 \\ -\sin \pi x, 1 < x < 3/2 \end{cases}$

$I = \int_0^1 \sin \pi x\,dx + \int_1^{3/2} -\sin \pi x\,dx = -\left[\frac{\cos \pi x}{\pi}\right]_0^1 + \left[\frac{\cos \pi x}{\pi}\right]_1^{3/2} = \frac{3}{\pi}.$

43. $\int_0^1 |\cos \pi x|\,dx = I$, say. $|\cos \pi x| = \begin{cases} \cos \pi x, 1 < x < 1/2 \\ -\cos \pi x, 1/2 < x < 1 \end{cases}$

Now, proceeding as in Qn 42. $I = \frac{2}{\pi}$.

44. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\sin x|\,dx = I$. $I = \int_{-\pi/2}^{0} -\sin x\,dx + \int_0^{\pi/2} \sin x\,dx = [\cos x]_{-\pi/2}^{0} - [\cos x]_0^{\pi/2} = 2$

45. $\int_0^{\pi} |\cos x|\,dx$ ***DO IT YOURSELF.*** [ANS : 2]

46. $\int_{-2}^{0} [x]\,dx$ ***DO IT YOURSELF.*** [ANS : - 3].

47. $\int_0^{\frac{3}{2}} [x]\,dx = \int_0^1 [x]\,dx + \int_1^{3/2} [x]\,dx = \int_0^1 0\,dx + \int_1^{3/2} 1\,dx = \frac{1}{2}$.

48. $\int_{-1}^{1} [x]\,dx = \int_{-1}^{0} [x]\,dx + \int_0^1 [x]\,dx = \int_{-1}^{0} -1\,dx + \int_0^1 0\,dx = -1$.

49. $\int_{-1}^{-1/2} \frac{1}{|x|}\,dx$ ***DO IT YOURSELF.*** [ANS : - log 2].

50. $\int_{-1}^{1} x^3\,dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0$.

51. $\int_{-\frac{\pi}{2}}^{\pi} \sin x\,dx = \{-\cos x\}_{-\pi/2}^{\pi} = 1$.

52. $\int_{-\pi}^{\pi} \cos x\,dx$ ***DO IT YOURSELF.*** [ANS : 0].

53. $\int_{-1}^{2} x|x|\,dx = \int_{-1}^{0} -x^2\,dx + \int_0^2 x^2\,dx = \frac{7}{3}$.

{ When x < 0 , |x| = - x and hence x|x| = - x^2; when x > 0, |x| = x and so x|x| = x^2}

54. $\int_{-1}^{1} |x^2|\,dx = \int_{-1}^{1} x^2\,dx = \frac{2}{3.}$ { $|x^2| = x^2$ always. i.e., for x < 0 and $x \geq 0$}.

55. $\int_{-1}^{1} |x^3|\,dx = \int_{-1}^{0} -x^3\,dx + \int_0^1 x^3\,dx = \frac{1}{2}$.

{ When x < 0 , $x^3 < 0$ and when $x \geq 0$, $x^3 \geq 0$ }.

56. $\int_{-3}^{-2} |x^3|\,dx.$ ***DO IT YOURSELF.*** [ANS : 65/4].

57. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \tan^2 x\,dx = 2\int_0^{\pi/4} \tan^2 x\,dx = 2\int_0^{\pi/4} \sec^2 x - 1\,dx$

$= 2\{\tan x - x\}_0^{\pi/4} = 2\left(1 - \frac{\pi}{4}\right)$

58. $\int_{-3}^{-2} \frac{1}{x}\,dx = \{\log|x|\}_{-3}^{-2} = \log 2 - \log 3 = \log\left(\frac{2}{3}\right)$.

59. $\int_{-2}^{2} |1 - x|\,dx = \int_{-2}^{1} (1 - x)\,dx + \int_1^2 -(1 - x)\,dx = 5$.

60. $\int_{-1}^{1} \left[\frac{x}{2}\right]\,dx = \int_{-1}^{0} \left[\frac{x}{2}\right]\,dx + \int_0^1 \left[\frac{x}{2}\right]\,dx = \int_{-1}^{0} -1\,dx + \int_0^1 0\,dx = 1$.

61. $\int_{-1}^{1}(x-[x])\,dx = \int_{-1}^{0} x - [x]\;dx + \int_{0}^{1} x - [x]\,dx.$

$$= \int_{-1}^{0} x + 1\;dx + \int_{0}^{1} x - 0\,dx = 1.$$

62. $\int_{1}^{3/2} [x^2]\;dx = \int_{1}^{\sqrt{2}}[x^2]\;dx + \int_{\sqrt{2}}^{3/2}[x^2]\;dx = \int_{1}^{\sqrt{2}} 1\,dx + \int_{\sqrt{2}}^{3/2} 2\,dx$

$$= \{x\}_{1}^{\sqrt{2}} + \{2x\}_{\sqrt{2}}^{3/2} = \sqrt{2} - 1 + 3 - 2\sqrt{2} = 2 - \sqrt{2}.$$

63. $\int_{-1}^{1} x|x|\;dx = \int_{-1}^{0} x|x|\;dx + \int_{0}^{1} x|x|\;dx = \int_{-1}^{0} -x^2\;dx + \int_{0}^{1} x^2\;dx = 0.$

64. $\int_{-1}^{1} x[x]\;dx = \int_{-1}^{0} x[x]\;dx + \int_{0}^{1} x[x]\;dx = \int_{-1}^{0} -x\,dx + \int_{0}^{1} 0\,dx$

$$= -\left\{\frac{x^2}{2}\right\}_{-1}^{0} = \frac{1}{2}.$$

65. $\int_{0}^{2\pi} \sin x\,dx = \{-\cos x\}_{0}^{2\pi} = 0.$

66. $\int_{0}^{2\pi} \cos x\,dx = \{\sin x\}_{0}^{2\pi} = 0.$

67. $\int_{-\pi/4}^{\pi/4} \tan x\,dx = 0$, *since* $f(x) = \tan x$ *is an odd function.*

68. ***DO IT YOURSELF.*** [ANS : 4/3]

69. ***DO IT YOURSELF.*** [ANS : 2]

70. ***DO IT YOURSELF.*** [ANS : 4/3]

71. $\int_{0}^{\frac{\pi}{2}} \sin^2 x\,dx = \int_{0}^{\pi/2} \frac{1-\cos 2x}{2} = \frac{1}{2}\left\{x - \frac{\sin 2x}{2}\right\}_{0}^{\pi/2} = \frac{\pi}{4}.$

72. $\int_{0}^{\pi} \sin |x| dx.$ ***DO IT YOURSELF.*** [ANS : 2].

73. $\int_{0}^{\pi} \cos |x| dx.$ ***DO IT YOURSELF.*** [ANS : 0].

74. $\int_{-1}^{1}(1-|x|)\,dx = \int_{-1}^{0} 1 + x\;dx + \int_{0}^{1} 1 - x\;dx$

$$= \left\{x + \frac{x^2}{2}\right\}_{-1}^{0} + \left\{x - \frac{x^2}{2}\right\}_{0}^{1} = 1.$$

Can also be done by taking $\int_{-1}^{1} 1\;dx - \int_{-1}^{1} |x|\;dx.$

75. $\int_{-a}^{a}(a-|x|)\,dx = \int_{-a}^{0} a + x\ dx + \int_{0}^{a} a - x\ dx$

$$= \left\{ax + \frac{x^2}{2}\right\}_{-a}^{0} + \left\{ax - \frac{x^2}{2}\right\}_{0}^{a} = a^2.$$

76. $\int_{0}^{2} |2x - 1|\,dx = \int_{0}^{1/2} |2x - 1|\,dx + \int_{1/2}^{2} |2x - 1|\,dx$

$$= \int_{0}^{1/2} -(2x - 1)\,dx + \int_{1/2}^{2} 2x - 1\,dx$$

$$= \{x - x^2\}_{0}^{1/2} + \{x^2 - x\}_{1/2}^{2} = 2.$$

77. $\int_{1}^{e} \frac{1}{x}\,dx = \{\log|x|\}_{1}^{e} = \log e - \log 1 = 1.$

78. $\int_{0}^{1} \frac{x^3+8}{x+2}\,dx = \int_{0}^{1} \frac{(x+2)(x^2-2x+4)}{x+2}\,dx = \frac{10}{3}.$

79. $\int_{0}^{3} |1 - x^2|\,dx = \int_{0}^{1} |1 - x^2|\,dx + \int_{1}^{3} |1 - x^2|\,dx$

$$= \int_{0}^{1} 1 - x^2\ dx + \int_{1}^{3} -(1 - x^2)\,dx = \frac{22}{3}.$$

80. $\int_{0}^{2} |x^3 - 1|\,dx = \int_{0}^{1} |x^3 - 1|\,dx + \int_{1}^{2} |x^3 - 1|\,dx$

$$= \int_{0}^{1} -(x^3 - 1)\,dx + \int_{1}^{2}(x^3 - 1)\,dx = \frac{15}{4}.$$

81. $\int_{-1}^{1} |x - x^2|\,dx = \int_{-1}^{0} |x - x^2|\,dx + \int_{0}^{1} |x - x^2|\,dx.$

$$= \int_{-1}^{0} -(x - x^2)\,dx + \int_{0}^{1}(x - x^2)\,dx = 1.$$

82. $\int_{-4}^{5} f(x)\,dx = \int_{-4}^{3} 9\,dx + \int_{3}^{5} x^2\,dx = \frac{287}{3}.$

83. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\sin^3 x \cos x + \cos x \sin x)dx = I, say.$

$f(x) = \sin^3 x \cos x + \cos x \sin x$

$= \cos x(\sin^3 x + \sin x)$ is an odd function. So, $I = 0$.

END OF CHAPTER 7

VIII. APPLICATIONS OF INTEGRATION

VERY SHORT ANSWER TYPE

1. Find the area bounded by $y = x$, the x-axis and the lines $x = -2$ and $x = 1$.
2. Find the area bounded by $y = x^2$, the y – axis between $y = 0$ and $y = 2$.
3. Find the area bounded by $y^2 = x^3$, the x – axis and between the lines $x = 0$ and $x = 1$.
4. Find the area bounded by $y = x^3$, the x-axis and between $x = -1$ and $x = 1$.
5. Find the area bounded by $y = |x|$, the x-axis and between $x = -2$ and $x = 2$.
6. Find the area bounded by $y = \sqrt{x}$, the y-axis and between $y = 0$ and $y = 1$.
7. Find the area bounded by $y = \frac{1}{x}$,the x-axis and between $x = 1$ and $x = 3$.
8. Find the area bounded by $y = \frac{1}{x}$,the x-axis and between $x = -2$ and $x = -1$.
9. Find the area bounded by $y = e^x$, the x-axis and between the lines $x = -2$ and $x = 1$.
10. Find the area bounded by $y = e^{-x}$, the x-axis and between the lines $x = 0$ and $x = 1$.
11. Using integration, find the area bounded by $x = 4$, $y = 2$,the x-axis and the lines $x = 0$ and $x = 4$.
12. Find the area bounded by $y = -|x|$, the x-axis and between $x = 1$ and $x = 2$.
13. Find the area bounded by the curve $y = \sin x$, the x-axis and between $x = \pi$ and $x = 2\pi$.
14. Find the area bounded by the curve $y = \sin x$, the x-axis and between

$x = 0$ and $x = \pi$.

15. Find the area bounded by the curve $y = \cos x$, the x-axis and between $x = 0$ and $x = \pi/2$.

16. Find the area bounded by the curve $y = \tan x$, the x-axis and between $x = 0$ and $x = \pi/4$.

17. Find the area bounded by the line $x + y = 1$, and the co-ordinate axes.

18. Find the area bounded by the line $x + y = 1$, the x-axis and the lines $x = 1$ and $x = 2$.

19. Find the area of the region bounded by $y = \log x$, the y-axis and the lines $y = -2$ and $y = -1$.

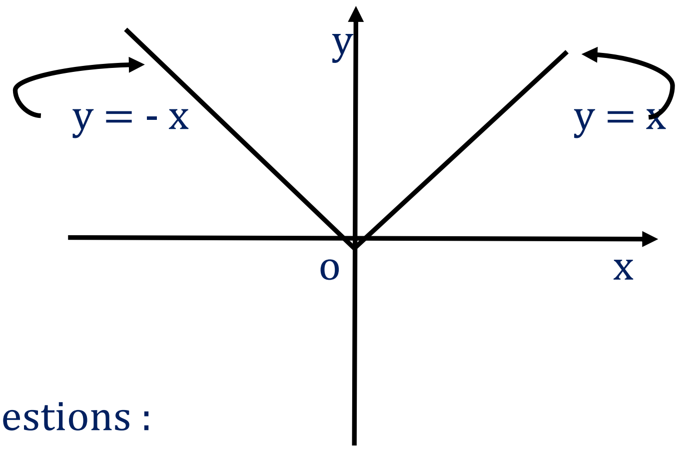
20. Find the area bounded by the line $x - y = 1$, and the co-ordinate axes.

VIII. APPLICATIONS OF INTEGRALS

CASE STUDY BASED MCQS

I. The function $y = f(x) = |x|$ defined by $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ is called the modulus function or absolute value function. The graph consists of two straight lines forming a V – shape. It is continuous everywhere but not differentiable at the origin.

The graph is as shown :



Now, answer the following questions :

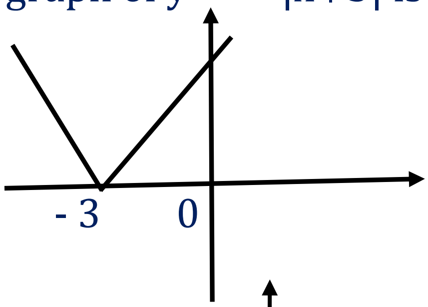
1. $\int_{-6}^{0} -|x+3|\,dx$ =

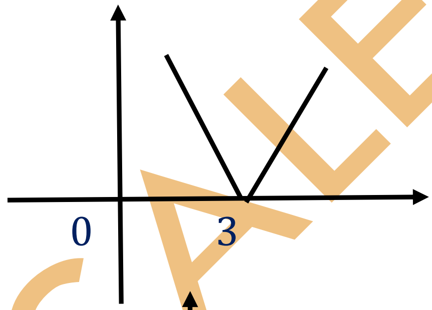
a) 9 b) 18 c) -9 d) -18

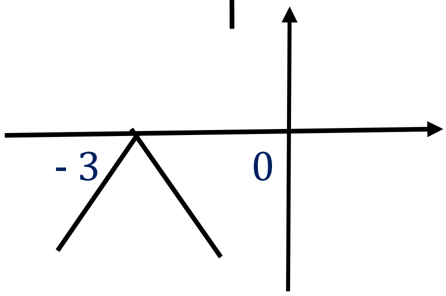
2. $\int_{0}^{6} -|x+3|\,dx$ =

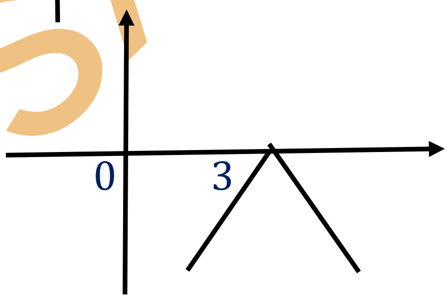
a) 9 b) -18 c) 0 d) -9

3. The graph of y = − |x+3| is

a) 

b) 

c) 

d) 

4. The area bounded by y = -|x+3|, the x-axis between x = -6 and x = 0 is the same as

a) $\int_{0}^{6} -|x+3|\,dx$

b) $\int_{-6}^{-3} |x+3|\,dx + \int_{-3}^{0} |x+3|\,dx$

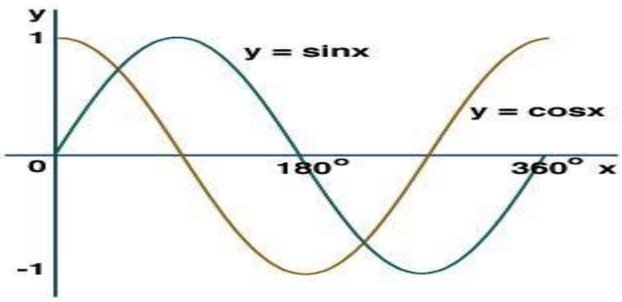
c) $\left|\int_{-6}^{0} -|x+3|\,dx\right|$

d) $\int_{0}^{3} |x+3|\,dx + \int_{3}^{6} |x+3|\,dx$

5. The area mentioned in Question 4 is

a) 18 b) 9 c) 0 d) 12

II. The graphs of the y = sin x and y = cos x are given in the diagram.



Now, answer the following questions:

1. The area bounded by $y = \cos x$, the x-axis between $x = 0$ and $x = \frac{\pi}{4}$ is

a) ½ b) $\sqrt{2}$ c) $\frac{1}{\sqrt{2}}$ d) $2\sqrt{2}$

2. The area bounded by $y = \sin x$, the x-axis between $x = 0$ and $x = \frac{\pi}{4}$ is

a) ½ b) $\sqrt{2} - 1$ c) $1 - \frac{1}{\sqrt{2}}$ d) $2\sqrt{2}$

3. The area bounded by $y = \sin x$, the x-axis between $x = 0$ and $x = \frac{\pi}{2}$ is

a) 1 b) $\sqrt{2}$ c) 2 d) $2\sqrt{2}$

4. The area bounded by $y = \cos x$, the x-axis between $x = 0$ and $x = \frac{\pi}{2}$ is

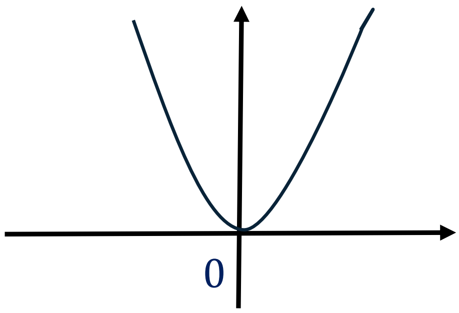
a) ½ b) $\sqrt{2}$ c) 0 d) 1

5. The area bounded by $y = \sin x$ and $y = \cos x$, between $x = 0$ and $x = \frac{\pi}{4}$ is

a) $2\sqrt{2}$ - 2 b) $\sqrt{2}$ - 1 c) $2 - \frac{1}{\sqrt{2}}$ d) $2\sqrt{2}$ - 2

III. The equation $x^2 = 4ay$ is a parabola which is open upward, has y-axis as its axis and vertex at the origin.

The curve is as shown .



Now, answer the following questions :

1. The area bounded by the parabola $y = x^2$,the x-axis and the lines $x = 0$, $x = 1$ is

a) ½ b) 2/3 c) 1/3 d) 1/6

2. The area bounded by the parabola $y = x^2$,the y-axis and the lines $y = 0$, $y = 2$ is

a) $\frac{\sqrt{2}}{3}$ b) 2/3 c) $\frac{4\sqrt{2}}{3}$ d) 4/3

3. The area bounded by the parabola $y = x^2$ and the lines $y = 0$, $y = 2$ is

a) $\frac{\sqrt{2}}{3}$ b) $\frac{3\sqrt{3}}{2}$ c) $\frac{4\sqrt{2}}{3}$ d) $\frac{8\sqrt{2}}{3}$

4. If the area bounded by the parabola $y = x^2$, the x-axis and the lines $x = 0$, $x = a$ is 9 sq. units, the value of "a" is

a) 2 b) 3 c) 1/3 d) 1/2

5. The area bounded by the parabola $y = x^2$ and the line $y = x$ is

a) ½ b) 2/3 c) 1/3 d) 1/6.

IV. The line $y = mx$ passes through the origin. As special cases, the lines $y = x$ and $y = - x$ are straight lines both passing through the origin, equally inclined to the coordinate axes.

Now, answer the questions that follow :

1. The area bounded by $y = x$, the x-axis and the lines $x = -2$ and $x = 2$ is

a) 4 b) 8 c) 6 d) 5.5

2. The area of the triangle formed by $y = x$, $y = -x$ and the line $x = 2$ is

a) 4 b) 8 c) 6 d) 10

3. The area bounded by the line $y = 3x$, x-axis between $x = - 2$, $x = 0$ is

a) 4 b) 8 c) 6 d) 9

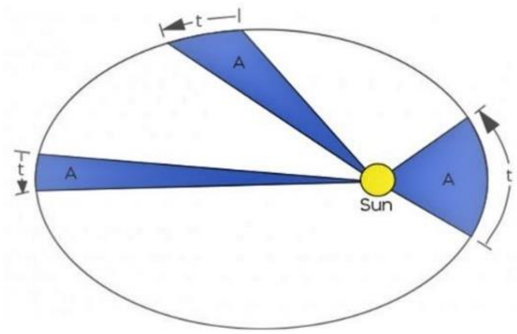
4. The area between $y = 3x$, $y = x$ and between $x = 0$ and $x = 2$ is

a) 4 b) 8 c) 6 d) 7

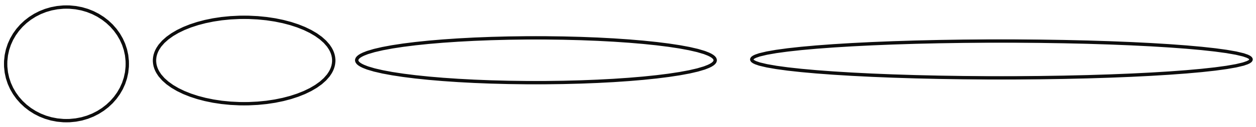
5. The area bounded by the line $y = 3x$, y – axis, between $y = 0$, $y = 3$ is

a) 7/2 b) 6 c) 5/3 d) 3/2 .

V_ The KEPLER'S SECOND LAW of planetary motion states that the radius vector joining the sun and a planet sweeps out equal areas in equal intervals of time.



The orbits of planets and satellites(natural or artificial) are elliptical while for some artificial satellites, they are circular.



Recall, an ellipse is a conic with eccentricity e < 1. e =0, for a circle, And as e approaches 1, the ellipse becomes flat. Refer figs., above.

Now, answer the following questions :

1. The parametric equations of the ellipse $25x^2 + 36y^2 = 900$ *are*

   a) $x = 5\cos\theta\,, y = 6\sin\theta$ b) $x = 6\cos\theta\,, y = 5\sin\theta$

   c) $x = 6\sin\theta\,, y = 5\cos\theta$ d) $x = 6\sec\theta\,, y = 5\tan\theta$

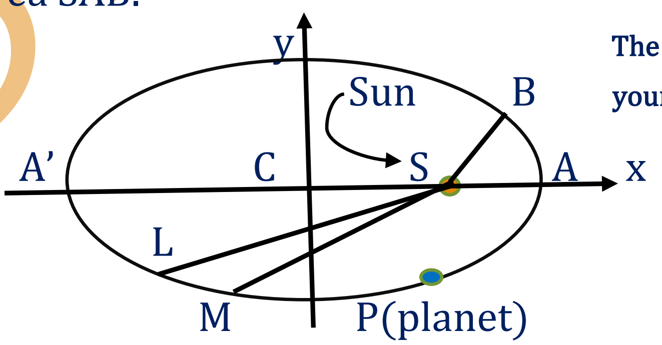
2. The cartesian equation of the ellipse whose parametric equations are $x = \cos\theta\,,\ 2y = 3\sin\theta\,, 0 \le \theta < 2\pi$ is

   a) $\frac{x^2}{4} + \frac{4y^2}{9} = 1$ b) $\frac{x^2}{1} + \frac{y^2}{9} = 1$

   c) $x^2 + \frac{4y^2}{9} = 1$ d) $\frac{x^2}{1} + \frac{9y^2}{4} = 1.$

Some planet P, moves around the sun in an elliptical orbit whose equation is $\frac{x^2}{16} + \frac{y^2}{9} = 1$. Read the Kepler's second law.

Area SLM = Area SAB.



The axes are drawn to help your understanding . . .

3. It is found that, corresponding to A and B, the values of θ are 0 *and* $\frac{\pi}{4}$. The cartesian coordinates of A and B are

a) $(0,4),(4\sqrt{2},\frac{1}{\sqrt{2}})$ b) $(4,0),\left(\frac{4}{\sqrt{2}},\frac{3}{\sqrt{2}}\right)$

c) $(4,0),(2\sqrt{2},\frac{1}{\sqrt{2}})$ d) $(3,0),(2\sqrt{2},\frac{3}{\sqrt{2}})$

4. The eccentricity e of the orbit of the planet P is

a) $5/4$ b) $\sqrt{7}/3$ c) $\sqrt{7}/4$ d) $5/3$.

5. For an experimental study of the orbit it is necessary to find the area swept out by the planet over a period of one month(say, in moving from L to M). By Kepler's law, area SLM = area SAB. The area of segment SAB is given by

a) $\int_{\sqrt{7}}^{3/\sqrt{2}} y_1\,dx + \int_{3/\sqrt{2}}^{4} y_2\ dx$ b) $\int_{\sqrt{7}}^{4/\sqrt{2}} y_1\,dx + \int_{4/\sqrt{2}}^{4} y_2\ dx$

c) $\int_{5/3}^{4/\sqrt{2}} y_1\,dx + \int_{4/\sqrt{2}}^{4} y_2\ dx$ d) $\int_{\sqrt{7}/4}^{4/\sqrt{2}} y_1\,dx + \int_{4/\sqrt{2}}^{4} y_2\ dx$.

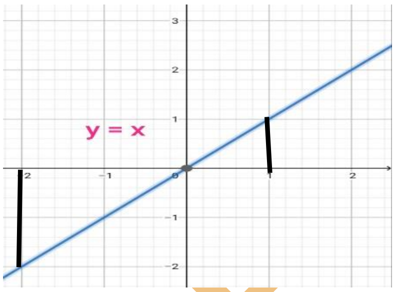
[NOTE : Here, y_1 is the y from the equation of line SB and y_2 is the y from the equation of ellipse.]

VIII. APPLICATIONS OF INTEGRALS – SOLUTIONS

1. Area $= \left|\int_{-2}^{0} y\,dx\right| + \int_{0}^{1} y dx$, y = x

$= \left|\int_{-2}^{0} x\,dx\right| + \int_{0}^{1} x\,dx$

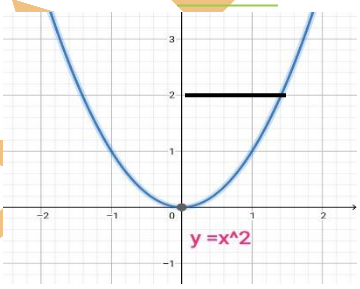
$= \left[\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{1} = 3/2.$



2. Area $= \int_{0}^{2} x\,dy = \int_{0}^{2} \sqrt{y}\,dy$, y = x²

$= \frac{2}{3}\left[y^{3/2}\right]_{0}^{2} = \frac{2}{3}\cdot 2^{3/2}$

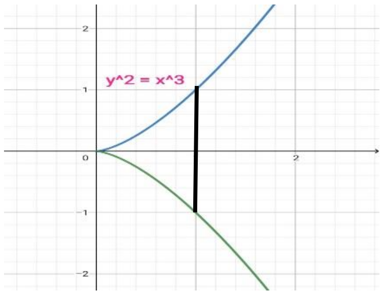
$= \frac{2}{3}\cdot 2\sqrt{2} = \frac{4\sqrt{2}}{3}.$



3. Area $= 2\int_{0}^{1} y\,dx$, $y = x^{3/2}$

$= 2\int_{0}^{1} x^{3/2}\,dx$

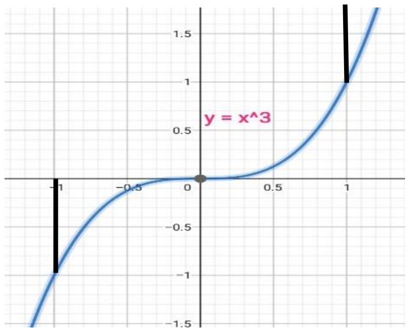
$= 2\cdot\frac{2}{5}\cdot\left[x^{5/2}\right]_{0}^{1} = \frac{4}{5}.$



4. Area $= \left|\int_{-1}^{0} y\,dx\right| + \int_{0}^{1} y\,dx$, y = x³

$= \left|\int_{-1}^{0} x^3\,dx\right| + \int_{0}^{1} x^3\,dx$

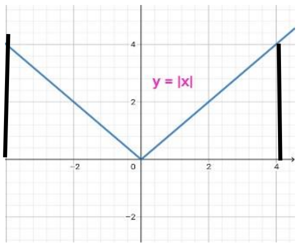
$= \left|-\left[\frac{x^4}{4}\right]_{-1}^{0}\right| + \left[\frac{x^4}{4}\right]_{0}^{1} = 1/2.$



5. Area $= \int_{-2}^{0} y\,dx + \int_{0}^{2} y\,dx$, y = - x, y = x

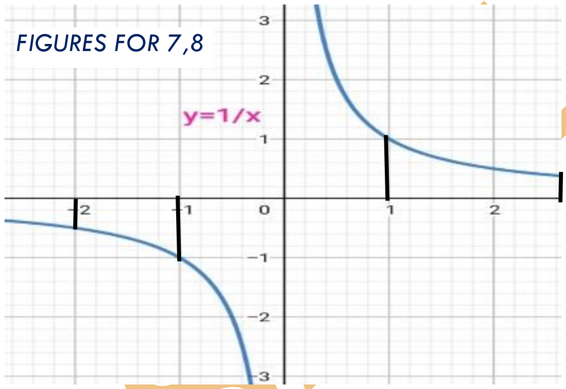
$= \int_{-2}^{0} \{-x\}\,dx + \int_{0}^{2} x\,dx$

$= -\left[\frac{x^2}{2}\right]_{-2}^{0} + \left[\frac{x^2}{2}\right]_{0}^{2} = 4.$



6. Area = $\int_0^1 x\, dy$, $y = \sqrt{x}$ *or* $y^2 = x$

$= \left[\frac{y^3}{3}\right]_0^1 = \frac{1}{3}$.



7. Area = $\int_1^3 y\, dx$, $y = 1/x$

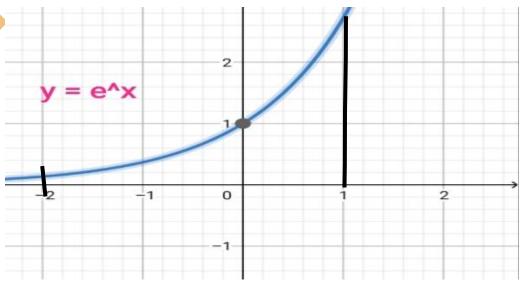
$= \int_1^3 \frac{1}{x}\, dx$

$= [\log |x|\,]_1^3 = \log 3$.

8. Area = $\left|\int_{-2}^{-1} y\, dx\right|$, $y = 1/x$

$= |[\log |x|\,]_{-2}^{-1}|$

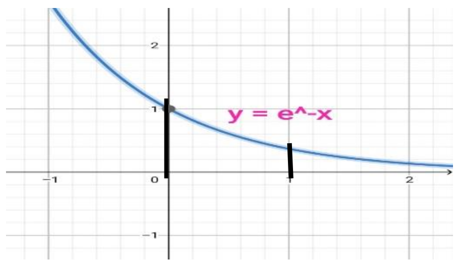
$= \log 2$.



9. Area = $\int_{-2}^{1} y\, dx$, $y = e^x$

$= [e^x]_{-2}^{1} = e^1 - e^{-2}$

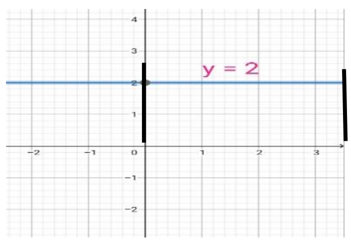
$= \frac{e^3 - 1}{e^2}$.



10. Area = $\int_0^1 y\, dx$, $y = e^{-x}$

$= -[e^{-x}]_0^1$

$= 1 - \frac{1}{e}$.



11. Area = $\int_0^4 y\, dx$, $y = 2$

$= \int_0^4 2\, dx$

$= \{2x\}_0^4 = 8$.

12. Area $= \left| \int_1^2 y\,dx \right|$, y = - |x|

$= \left| - \left[\frac{x^2}{2}\right]_1^2 \right|$

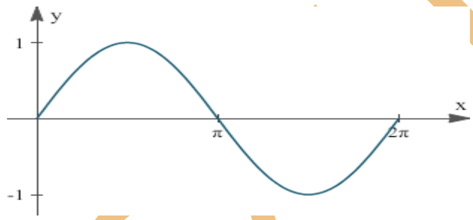
$= 3/2.$

REFER CHAPTER VI, FOR THE GRAPH.

13. Area $= \left| \int_\pi^{2\pi} y\,dx \right|$, y = sin x

$= \left| -\{\cos x\}_\pi^{2\pi} \right|$

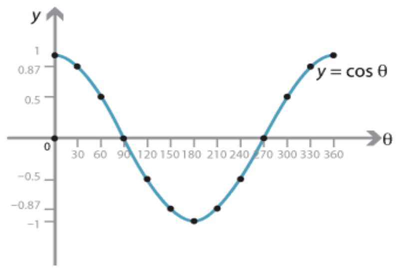
$= 2.$



15. Area $= \int_0^{\frac{\pi}{2}} y\,dx$, y = cos x

$= \{\sin x\}_0^{\frac{\pi}{2}}$

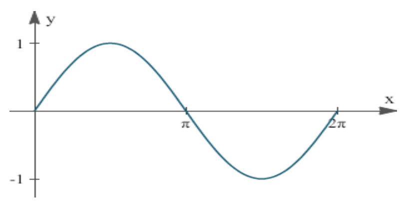
$= 1.$



14. Area $= \int_0^\pi y\,dx$, y = sin x

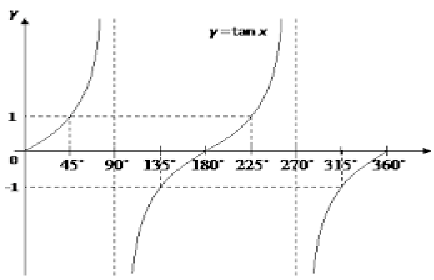
$= -\{\cos x\}_0^\pi$

$= 2.$



16. Area $= \int_0^{\pi/4} y\,dx$, y = tan x

$= \{\log|\sec x|\}_0^{\pi/4} = \frac{1}{2}\log 2$.



17. Area $= \int_0^1 y\,dx$, y = 1 – x

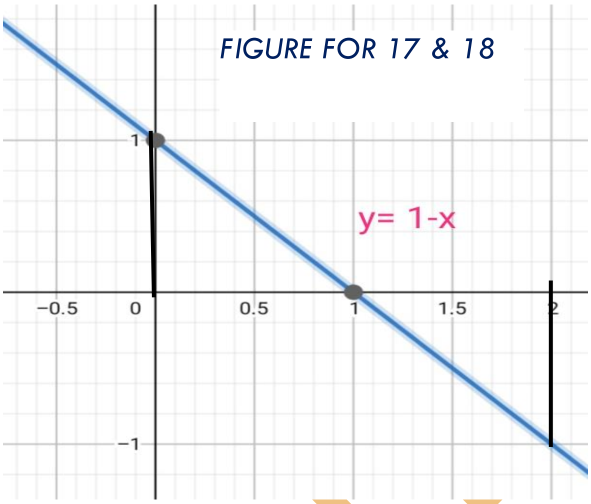
$= \left\{x - \frac{x^2}{2}\right\}_0^1$

$= ½.$

18. Area = $\left|\int_1^2 y\,dx\right|$, y = 1 – x

$= \left|\left\{x - \frac{x^2}{2}\right\}_1^2\right|$

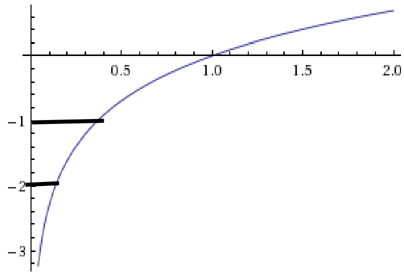
$= ½.$



19. Area = $\int_{-2}^{-1} x\,dy$, y = $\log_e x$

$= [e^y]_{-2}^{-1}$

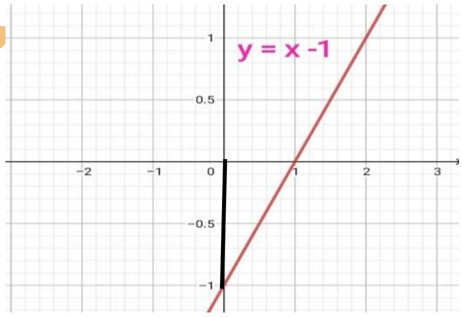
$= e^{-1} - e^{-2} = \frac{e-1}{e^2}.$



20. Area = $\left|\int_0^1 y\,dx\right|$, y = x - 1

$= \left|\left[\frac{x^2}{2} - x\right]_0^1\right|$

$= 1/2.$



VIII - SOLUTIONS TO CASE STUDIES

I. 1. $\int_{-6}^{0} -|x+3|\,dx = \int_{-6}^{-3}(x+3)\,dx + \int_{-3}^{0} -(x+3)\,dx$

$= \left[\frac{x^2}{2} + 3x\right]_{-6}^{-3} - \left[\frac{x^2}{2} + 3x\right]_{-3}^{0} = -9.$

2. $\int_0^6 -|x+3|\,dx = \int_0^6 -(x+3)\,dx = -\left[\frac{x^2}{2} + 3x\right]_0^6 = -9.$

3. C)

4. Area bounded by y = - |x + 3|, x-axis between x = - 6, x = 0 is

Area = $\left|\int_{-6}^{0} -|x+3|\,dx\right|$

5. The area is 9 square units.

II. 1. $\int_0^{\pi/4} \cos x \, dx = \{\sin x\}_0^{\pi/4} = \frac{1}{\sqrt{2}}.$

2. $\int_0^{\pi/4} \sin x \, dx = \{-\cos x\}_0^{\pi/4} = -\{\cos x\}_0^{\pi/4} = -(\frac{1}{\sqrt{2}} - 1) = 1 - \frac{1}{\sqrt{2}}.$

3. $\int_0^{\pi/2} \sin x \, dx = \{-\cos x\}_0^{\pi/2} = -(0 - 1) = 1.$

4. $\int_0^{\pi/2} \cos x \, dx = \{\sin x\}_0^{\pi/2} = 1 - 0 = 1.$

5. $\int_0^{\pi/4} (\cos x - \sin x) \, dx = \frac{1}{\sqrt{2}} - (1 - \frac{1}{\sqrt{2}}) = \sqrt{2} - 1.$

III. 1. $\int_0^1 x^2 \, dx = 1/3.$

2. Area = $\int_0^2 x \, dy = \int_0^2 \sqrt{y} \, dy$, $y = x^2$

$= \frac{2}{3}\left[y^{3/2}\right]_0^2 = \frac{2}{3} \cdot 2^{3/2} = \frac{2}{3} \cdot 2\sqrt{2} = \frac{4\sqrt{2}}{3}.$

3. Area = 2 x $\int_0^2 x \, dy = 2 \text{ x } \frac{4\sqrt{2}}{3} = \frac{8\sqrt{2}}{3}.$

4. $\int_0^a y \, dx = 9 \Rightarrow \int_0^a x^2 \, dx = \left[\frac{x^3}{3}\right]_0^a = 9 \Rightarrow a = 3.$

5. $\int_0^1 (\underbrace{y}_{line} - \underbrace{y}_{parabola})dx = \int_0^1 (x - x^2)dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1/2 - 1/3 = 1/6.$

IV. 1. Area = $\int_{-2}^2 y \, dx = \left|\int_{-2}^0 y \, dx\right| + \int_0^2 y \, dx = \left|\left[\frac{x^2}{2}\right]_{-2}^0\right| + \left[\frac{x^2}{2}\right]_0^2 = 4.$

2. Area = 2x Area below y = x between x = 0 and x = 2

$= 2 \text{ x } \int_0^2 y \, dx = 2 \text{ x } 2 = 4.$

3. Area = $\int_0^2 y \, dx = \int_0^2 3x \, dx = 6.$

4. Area = $\int_0^2 \{y_1 - y_2\} \, dx = \int_0^2 (3x - x)dx = 4.$

5. Area = $\int_0^3 x \, dy = \int_0^3 \frac{y}{3} \, dy = \frac{3}{2}.$

V. 1. b 2. c 3. b 4. c 5. b

END OF CHAPTER 8

IX. DIFFERENTIAL EQUATIONS

VERY SHORT ANSWER TYPE

1. State the order of the differential equation for which $y = ae^{b + cx}$ is a solution.

2. Write the degree of the differential equation $\left[1 + \left(\frac{d^2y}{dx^2}\right)^2\right]^{3/2} = \frac{dy}{dx}$.

3. Write the degree of the differential equation $\left[1 + \left(\frac{dy}{dx}\right)^2\right] = \sqrt{\frac{d^2y}{dx^2}}$.

4. Write the degree of the differential equation $1 + \left(\frac{dy}{dx}\right)^2 = x.\frac{dx}{dy}$.

5. Write the degree of the differential equation $\frac{d^2y}{dx^2} + \sqrt{x} = \left(\frac{dy}{dx}\right)^{1/3}$.

6. Write the degree of the differential equation $(1 + y')^2 = x + (y')^2$.

7. Write the degree of the differential equation $\sin\left(\frac{dy}{dx}\right) = x$.

8. Write the degree of the differential equation $y' + (y'')^2 = x(x + y'')^2$.

9. Write the degree of the differential equation $\left(\frac{dy}{dx}\right)^2 + x = \frac{dx}{dy} + x^2$.

10. Write the degree of the differential equation $y'' = (y'' - (y')^3)^{3/2}$.

11. Find the family of curves represented by the solution of the differential equation $y\frac{dy}{dx} + x = c$.

12. Find the order of the differential equation for which $y = A^3x + B$ is a solution, where A is the arbitrary constant.

13. Find the order of the differential equation for which $y = A^3x + A$ is a solution, where A is the arbitrary constant.

14. At any point (x,y) of a curve, the slope of the tangent is twice the slope

of the line segment joining the point of contact to the point (- 4,- 3). Formulate the differential equation for this.(You need not solve the differential equation).

15. Find the order of the differential equation of the family of circles in the third quadrant and touching the co-ordinate axes.
16. In a bank, the principal increases continuously at the rate of 8% per annum. Write the differential equation concerning the growth of the principal.
17. Find the order of the differential equation of all circles in the xy plane whose radius is 5 units.
18. Find the order of the differential equation of all circles in the xy plane whose radius is r units.
19. Find the order of the differential equation of all ellipses with centre at origin and semi major axis 6 units.
20. The velocity of a moving object at time t is given by v = 16 – 2t. Find the equation for the displacement s(t) , given that s(0) = 0.
21. Describe the nature of the family of curves represented by the solution of the differential equation $\frac{dy}{dx} = m$.
22. Solve : xdy + ydx = xy dx.
23. Solve : xdy + ydx = 0.
24. Solve : $\frac{dy}{dx} = c, given\ that\ y(1) = -2$.
25. Solve : $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$,
26. Solve : $\frac{dy}{dx} = e^{x+y}$.
27. Find the integrating factor of $\frac{dy}{dx} + \frac{2y}{x} = sin\ x$.
28. Find the integrating factor of $\frac{dy}{dx} - \frac{y}{xlogx} = 2x$.
29. Solve : $\frac{dy}{dx} + my = 0$.
30. Solve : $\sin\left(\frac{dy}{dx}\right) = 1$.

31. Solve : (x – y)dx = (x + y)dy.

32. Find the integrating factor of $(x+1)\frac{dy}{dx} - y = e^x(x+1)^2$.

33. Find the integrating factor of $(y+x)\frac{dx}{dy} = a^2$.

34. F+ind the integrating factor of $x\frac{dy}{dx} + 2\sqrt{x}\,y = 1 + x$.

35. Solve : dx + dy = 0, given that y(1) = 0.

36. Solve : $\left(\frac{dy}{dx}\right)^2 + 1 = 2x$.

37. Solve : $\frac{dy}{dx} - y = 0$.

38. Find the integrating factor of $\frac{dy}{dx} + \frac{y}{x} = e^x + y$.

39. Solve : $\log\left(\frac{dy}{dx}\right) = x - y$.

40. Solve : $\frac{dy}{dx} = (xy + 3y)^2$.

41. Find the integrating factor of $e^x\frac{dy}{dx} = 20 + 3e^x y$.

42. Find the integrating factor of $\frac{dy}{dx} = ysinx$.

43. Express $xy^2\frac{dy}{dx} = x^3 + y^3$ as a homogenous differential equation of the form $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$.

IN QUESTIONS 44 – 47, CLASSIFY THE D.E AS (i) SOLVABLE BY VARIABLES SEPERABLE (ii) LINEAR D.E (iii) HOMOGENOUS D.E.

44. $\frac{dy}{dx} = \frac{y}{x} + \frac{x}{y}$.

45. $(y - x)\frac{dx}{dy} = x$.

46. $\frac{dy}{dx} = \frac{y}{x} + \left(\frac{x}{y}\right)^2$.

47. $\frac{dy}{dx} - \frac{3}{x}y = \left(\frac{y}{x}\right)^2$.

48. Solve : $\frac{dy}{dx} = |x|$

49. Find the IF : $\frac{dy}{dx} - 4y = 8$.

50. Find the IF : $\sqrt{x}\,\frac{dy}{dx} = x + 3y$.

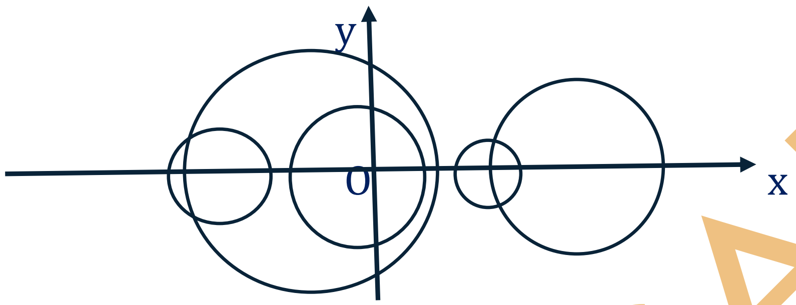
51. Find the IF : $x\frac{dy}{dx} + (5x + 2)y = \frac{20}{x}$

52. Find the IF : $2\sqrt{x}\,\frac{dy}{dx} + 3xy = 2xe^{-\sqrt{x}}$.

IX. DIFFERENTIAL EQUATIONS

CASE STUDY BASED MCQS

I. The diagram shows a family of circles all having centres on x-axis..



Now, answer the following questions.

1. The equation representing this family of circles is

a) $(x - h)^2 + (y - k)^2 = a^2$ b) $x^2 + y^2 = a^2$

c) $(x - h)^2 + y^2 = a^2$ d) $x^2 + (y - k)^2 = a^2$.

2. Let it be known that the differential equation representing this family of circles is $yy'' + (y')^2 + 1 = 0$. Then, the number of arbitrary constants in the general and particular solutions will be

a) 0, 1 b) 0, 2 c) 2, 1 d) 2, 0.

3. The degree of the differential equation is

a) 1 b) 2 c) 0 d) not defined.

4. The solution curve which passes through the point (5,0) and radius 5 is

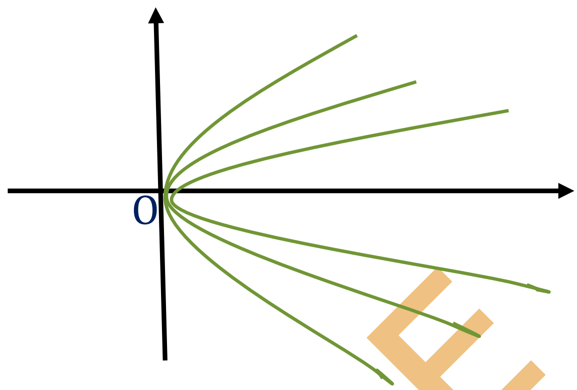
a) $x^2 + y^2 = 25$, b) $x^2 + y^2 - 10x = 0$

c) $x^2 + y^2 - 10y = 0$. d) $x^2 + y^2 - 20x = 0$.

5. The slope of the normal to the curve which passes through the point (1, -5) is

a) $\frac{1-x}{y}$ b) 0 c) not defined d) $\frac{y}{x-1}$.

II. The diagram shows a family of right open parabolas with vertex at the origin.



Now, answer the following questions.

1. The equation representing this family of parabolas is

   a) $y = 4ax^2$ b) $y^2 = 4ax$ c) $y^2 = - 4ax$ d) $y^2 = 4a(x - a)$

2. The order of the differential equation representing this family of parabolas is

   a) 1 b) 2 c) 0 d) not defined.

3. The equation of the parabola whose focus is at a distance of 5 units from the origin is

   a) $y^2 = 20x$ b) $x^2= 20y$ c) $y^2 = 5x$ d) $y^2 = 4(x - 5)$.

4. The slope of normal to the curve passing through (1,1) at the origin is

   a) 0 b) – 1 c) 1 d) not defined.

5. The point on the solution curve obtained in the above Qn.3, where the slope of the tangent is 1 is

   a) (10,5) b) (5,5) c) (5,10) d) (- 10,5).

III. Consider the differential equation $x\,dy + y\,dx = 0$.

Answer the following questions.

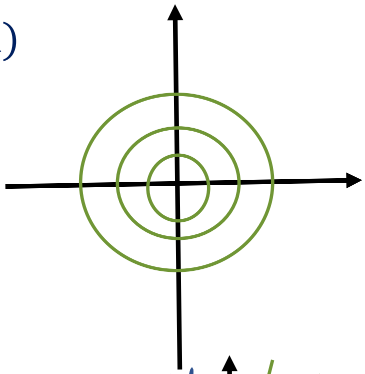
1. The order and degree of the differential equation are

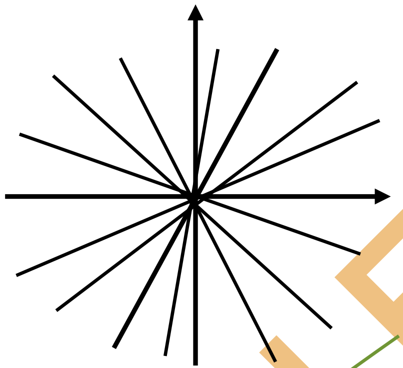
   a) 1,2 b) 2,2 c) 2,1 d) 1,1.

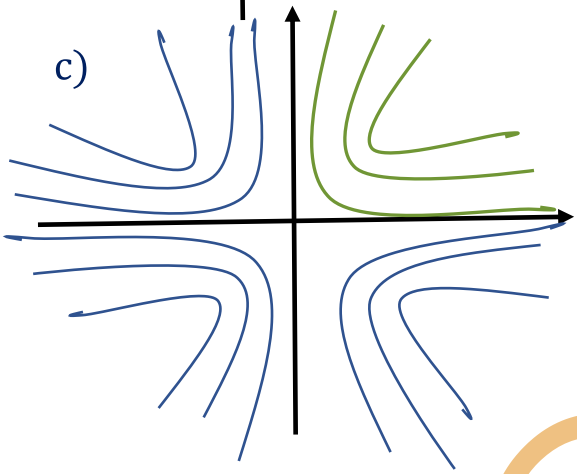
2. The general solution of the differential equation is

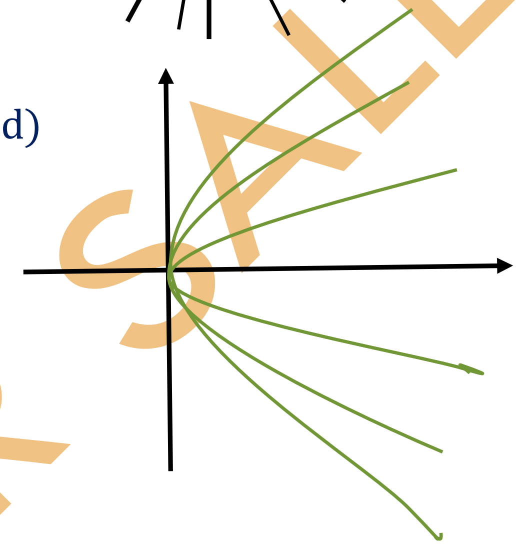
   a) $x = cy$ b) $x + y = c$ c) $xy = c$ d) $x^2 + y^2 = c$.

3. The family of curves representing the general solution of the given differential equation is

a) 

b) 

c) 

d) 

4. The particular solution when $y = 1$, $x = 1$ is

a) $y = x$ b) $x + y = 1$ c) $x^2 + y^2 = 1$ d) $xy = 1$.

5. The graph of the solution curve obtained in Qn. 4 is

a) decreasing   b) increasing

c) strictly decreasing   d) strictly increasing.

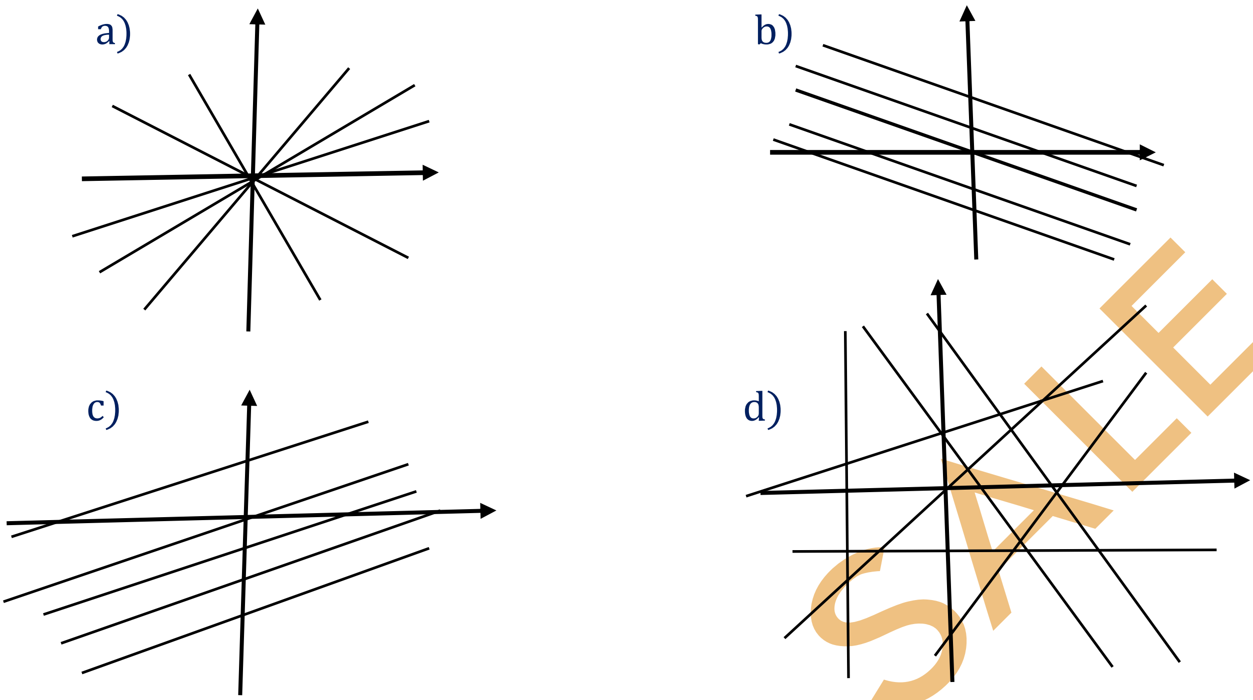
IV. Consider the differential equation $\sin^{-1}\left(\frac{dy}{dx}\right) = \frac{\pi}{2}$.

Now, answer the following questions :

1. The degree of the differential equation is

a) 1   b) 2   c) 0   d) not defined.

2. Which of the following family of lines is represented by the given differential equation ?



3. The solution curve passing through (1,1) is

a) y = x   b) y = x – 1   c) y = x + 1   d) x + y = 2

4. The line whose y-intercept is – 2 passes through the point

a) (3,1)   b) (0,2)   c) (1,3)   d) (- 4 , 2)

5. The family of lines which are perpendicular to the family of lines given by the differential equation is

a) y = - cx   b) x + y = c   c) x - y = c   d) y = - 2x + c

V. Let P denote the principal deposited with a bank which gives an interest of r % per year. Ram invests Rs. 1000 in a bank which pays 5% interest per year.

Now, answer the following questions.

1. The differential equation governing the growth of the principal is

a) $\frac{dP}{dt} = r\%$ b) $\frac{dP}{dt} = P$ c) $\frac{dP}{dt} = kt$ d) $\frac{dP}{dt} = kP.$

2. The general solution is given by

a) $P = ce^{kt}$ b) $P = e^{kt}$ c) $P = ce^{t}$ d) $P = ce^{-kt}$.

3. The time required for the principal Rs.1000 to double is (approximately) [HINT : Use $\log_e 2 = 0.6931$]

a) 11 years   b) 12 years   c) 14 years   d) 13 years.

4. What wil be the amount in 10 years ? [use $e^{0.5} = 1.648$]

a) Rs.1390   b) Rs. 1648   c) Rs.2718   d) Rs.1448.

5. What will be the amount in 10 years if r = 10% ? [use e ≈ 2.718]

a) Rs. 1648   b) Rs. 5436   c) Rs. 2718   d) Rs.3296.

VI. The Newton's law of cooling states that the rate of change of temperature of an object is proportional to the difference in the temperatures of the object and that of the surrounding medium.

Let T denote the temperature of the object at time t and S denote the temperature of the surrounding medium(say, room).

At 10.00 a.m, a cup of coffee at temperature 100° C is placed in a room whose temperature is 25° C. After 5 minutes, the temperature reduces to 80° C.

Now, answer the following questions :

1. The mathematical formulation of the Newton's law of cooling is

a) $\frac{dT}{dt} \propto T$ b) $\frac{dT}{dt} \propto T - S$ c) $\frac{dT}{T} \propto T - S$ d) none of these.

2. The general solution of the differential equation governing the temperature T is

a) $T = ce^{kt}$ b) $T = ce^{kt} - S$ c) $T = S + ce^{kt}$ d) none of these.

[NOTE : HERE, k IS THE CONSTANT OF PROPORTIONALITY AND c IS THE ARBITRARY CONSTANT].

3. The value of "c" is given by

a) 25 b) 100 c) 75 d) any real number.

4. The value of "k" is given by

a) $\frac{1}{5}\log_{10}\left(\frac{11}{15}\right)$ b) $5\log_{e}\left(\frac{11}{15}\right)$

c) $\log_{10}\left(\frac{11}{15}\right)$ d) $\frac{1}{5}\log_{e}\left(\frac{11}{15}\right)$.

5. The temperature of the coffee after a further period of 5 minutes is (approximately)

a) 60 °C b) 65 °C c) 63 °C d) 66 °C .

IX. DIFFERENTIAL EQUATIONS - SOLUTIONS

1. $y = ae^{b+cx} = ae^{b}e^{cx} = pe^{cx}$, where p , c are two arbitrary constants. The order is 2.

2. Squarring, $\left[1 + \left(\frac{d^2y}{dx^2}\right)^2\right]^3 = \left(\frac{dy}{dx}\right)^2$. Expanding the LHS using $(a+b)^3$ formula, there is a $\left[\left(\frac{d^2y}{dx^2}\right)^2\right]^3$ term. Therefore, the degree is 6.

3. Squarring on both sides, $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^2 = \frac{d^2y}{dx^2}$,Therefore, the degree is 1.

4. Rewriting, $1 + \left(\frac{dy}{dx}\right)^2 = x.\frac{1}{dy/dx}$, Cross-multipliction gives,

$\frac{dy}{dx} + \left(\frac{dy}{dx}\right)^3 = x$. Therefore, the degree is 3.

5. Cubing the equation, $\left[\frac{d^2y}{dx^2} + \sqrt{x}\right]^3 = \frac{dy}{dx}$, and expanding the LHS, there will be a term $\left(\frac{d^2y}{dx^2}\right)^3$, which implies that the degree is 3.

6. Expanding the LHS, we get $1 + 2y' + (y')^2 = x + (y')^2$. Cancelling out the $(y')^2$ term, the degree is 1.

7. Rewriting, $\left(\frac{dy}{dx}\right) = \sin^{-1} x$. Therefore, the degree is 1.

8. Expanding cannot cancel out the $(y'')^2$ term, so degree is 2.

9. Writing $\frac{dx}{dy}$ as $\frac{1}{dy/_{dx}}$ and then cross-multiplying, the equation becomes

$\left(\frac{dy}{dx}\right)^3 + x\frac{dy}{dx} = 1 + x^2\frac{dy}{dx}$. The degree is 3.

10. Squarring, $(y'')^2 = (y'' - (y')^3)^3$. The RHS expansion gives the term $(y'')^3$ and hence, the degree is 3.

11. Rewriting, $y\,dy + x\,dx = c\,dx$. Integrating, $\frac{y^2}{2} + \frac{x^2}{2} = cx + k'$

Multiplying by 2 and taking 2k' as k, the solution is $x^2 + y^2 - 2cx = k$, (where k is the arbitrary constant).

12. Though there are two constants A and B, it is given that A is the arbitrary constant. Therefore, the order is 1.

13. Since A is the only arbitrary constant, the order is 1.

14. We know that, Slope of the tangent = $\frac{dy}{dx}$.

Also, from given, Slope = 2 x slope of line joining point of contact and(-4,-3).

$= 2 \times \frac{y+3}{x+4}$

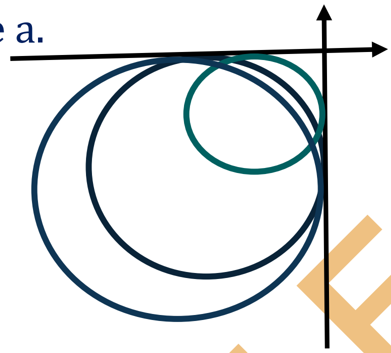
* Point of contact of tangent is (x,y).

* Slope = $\frac{y_2 - y_1}{x_2 - x_1}$

Therefore $\frac{dy}{dx} = 2.\frac{y+3}{x+4}$. is the required DE.

15. The diagram shows some members of the family of circles in the

third quadrant and touching the co-ordinate axes. If the co-ordinates of the centres are (- a,- a), then the radii will be a.



Hence, the equation of the family is

$(x + a)^2 + (y + a)^2 = a^2$, which involves one arbitrary constant.

Hence, the order of the DE will be 1.

16. Let P be the principal deposited with the bank. The rate of growth is Proportional to the the principal itself. i.e., $\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP$,

Where the constant of proportionality k = 8% (here).

Therefore, the DE is $\frac{dP}{dt} = \frac{8}{100}P$.

17. The equation of all circles in the xy-plane is $(x - h)^2 + (y - k)^2 = r^2$, where the centre is (h,k) and radius is r. Here, r = 5.

Therefore, the equation is $(x - h)^2 + (y - k)^2 = 5^2$, which contains two arbitrary constants viz., h and k(parameters). Hence, order is 2.

18. The equation of all circles in the xy-plane is $(x - h)^2 + (y - k)^2 = r^2$, where the centre is (h,k) and radius is r. There are 3 arbitrary constants h,k and r(parameters). Hence, the order is 3.

19. The equation of all ellipses with centre at origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a = semi-major axis , b = semi-minor axis. Given, a = 6 and so there is 1 arbitrary constant b(parameter). Hence, order is 1.

20. v = 16 – 2t. We know that velocity, v = $\frac{ds}{dt}$.

$\frac{ds}{dt} = 16 - 2t$

Seperating the variables,

ds = (16 – 2t) dt

Integrating, s(t) = 16t – t^2 + c

When t = 0, s = 0.

Sub, c = 0. ∴ The equation of motion is s(t) = 16t – t^2.

21. $\frac{dy}{dx} = m$. Solving, we get y = mx + c , which represents the family of straight lines in the xy- plane.

22. Given, xdy + ydx = xy dx.

That is d(xy) = xy dx

Seperating and integrating ,

$\int \frac{d(xy)}{xy} = \int dx$; logxy = x + c

Or, xy = e^{x+c} ; xy = $e^x e^c$; xy = k.e^x is the solution.

The product rule is

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

(Or) d(uv) = udv + vdu

23. x dy + y dx = 0 ; d(xy) = 0; integrating, xy = c is the solution.

24. dy = c dx ; integrating, $\int dy = \int c\ dx$; y = cx + k.

Given y = - 2 when x = 1. Sub, k = - 2 - c.

Therefore, the solution is y = cx – 2 - c.

[Note that, k is the arbitrary constant].

25. Seperating the variables and integrating, $\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$,

$c' + \tan^{-1} y = \tan^{-1} x$; $\tan^{-1} x - \tan^{-1} y = c'$. or $\frac{x-y}{1+xy} = c$.

26. $\frac{dy}{dx} = e^x e^y$, separating the variables and integrating,

e^{-y} dy = e^x dx ; solving, c – e^{-y} = e^x ; $e^x + e^{-y}$ = c is the solution.

27. P = 2/x. $\int P\ dx = \int \frac{2}{x}\ dx = 2\log x$, IF = $e^{\int P\,dx} = x^2$.

28. The given DE is $\frac{dy}{dx} - \frac{y}{x\log x} = 2x$. Here, P = $-\frac{1}{x\log x} = -\frac{1/x}{\log x}$.

$\int P\ dx = \int -\frac{1/x}{\log x}\ dx = -\log(\log x)$.

Therefore, IF = $e^{\int P\,dx} = e^{-\log(\log x)} = e^{\log(\log x)^{-1}} = \frac{1}{\log x}$.

29. $\frac{dy}{dx} = -my$, separating the variables, $\frac{dy}{y} = -m\,dx$. Integration gives

log y = - mx + c is the solution.

[OR, y = e^{-mx+c} ; y = $e^{-mx}.e^{c}$. Therefore, y = c' e^{-mx}].

30. $\sin\left(\frac{dy}{dx}\right) = 1$; $\frac{dy}{dx} = \sin^{-1}1 = \frac{\pi}{2}$.

dy = $\frac{\pi}{2}$ dx ; y = $\frac{\pi}{2}x + c$ is the solution.

31. (x – y)dx = (x + y)dy ⇒ xdx – ydx = xdy + ydy

⇒ xdx – ydy = xdy + ydx ⇒ xdx – ydx = d(xy). Integrating,

$\frac{x^2}{2} - \frac{y^2}{2} = xy + c'$ ⇒ $x^2 - y^2 - 2xy$ = c is the solution.

32. Given : $(x+1)\frac{dy}{dx} - y = e^x(x+1)^2$.

Dividing by (x + 1),eqn. becomes

$\frac{dy}{dx} - \frac{y}{x+1} = e^x(x+1)$. Here , P = $-\frac{1}{x+1}$.

$\int P\ dx = \int -\frac{1}{x+1}\ dx = -\log(x+1)$

Therefore, IF $= e^{\int P\ dx} = e^{-\log(x+1)} = e^{\log(x+1)^{-1}} = \frac{1}{x+1}$.

33. $(y+x)\frac{dx}{dy} = a^2$. Rewriting, $\frac{y+x}{a^2} = \frac{dy}{dx}$ (Or) $\frac{dy}{dx} - \frac{y}{a^2} = \frac{x}{a^2}$.

Here, P = $-1/a^2$. $\int P\ dx = -\int \frac{1}{a^2}\ dx = -\frac{x}{a^2}$

Therefore, IF $= e^{\int P\ dx} = e^{-x/a^2}$.

34. $x\frac{dy}{dx} + 2\sqrt{x}\ y = 1 + x$. dividing by x, $\frac{dy}{dx} + 2\frac{1}{\sqrt{x}}\ y = \frac{1+x}{x}$.

Here, P = $2\frac{1}{\sqrt{x}}$. $\int P\ dx = \int 2\frac{1}{\sqrt{x}}\ dx = 4\int \frac{1}{2\sqrt{x}}\ dx = 4\sqrt{x}$.

IF $= e^{\int P\ dx} = e^{4\sqrt{x}}$.

35. $dx + dy = 0 \Rightarrow$ integrating, $x + y = c$. Given, $y = 0$ when $x = 1$.

$\Rightarrow c = 1$. Therefore, the solution is $x + y = 1$.

36. $\frac{dy}{dx} = \sqrt{2x-1} \Rightarrow dy = \sqrt{2x-1}\ dx$; $y = \frac{1}{2}\cdot\frac{(2x-1)^{3/2}}{3/2} + c$.

37. $\frac{dy}{dx} = y \Rightarrow \frac{dy}{y} = dx \Rightarrow \log y = x + \log c \Rightarrow \log_e \frac{y}{c} = x$.

Changing to exponential form, $y = ce^x$ is the solution.

[$\log y = x + c$, can also be left as the answer].

38. $\frac{dy}{dx} + \frac{y}{x} = e^x + y \Rightarrow \frac{dy}{dx} + \frac{y}{x} - y = e^x$

$\Rightarrow \frac{dy}{dx} + \left(\frac{1}{x} - 1\right)y = e^x$; Linear in y. P $= \left(\frac{1}{x} - 1\right)$

$\int P\ dx = \int \frac{1}{x} - 1\ dx = \log x - x$.

$\text{IF} = e^{\int P\,dx} = e^{\log x\,-x} = e^{\log x}\cdot e^{-x} = xe^{-x}$.

39. $\log\left(\frac{dy}{dx}\right) = x - y. \Rightarrow \frac{dy}{dx} = e^{x-y} = e^{x}\cdot e^{-y}$, separating the variables

$e^y\,dy = e^x\,dx$ and integrating, $e^y = e^x + c$ is the solution.

40. $\frac{dy}{dx} = (xy+3y)^2$; $\frac{dy}{dx} = y^2(x+3)^2$

$\Rightarrow y^{-2}\,dy = (x+3)^2 dx$; $-\frac{1}{y} = \frac{(x+3)^3}{3} + c$.

41. the integrating factor of $e^x\frac{dy}{dx} = 20 + 3e^x y$.

Dividing by e^x and rewriting, $\frac{dy}{dx} - 3y = 20e^{-x}$. Here, P = - 3.

$\text{IF} = e^{\int p\,dx} = e^{\int -3\,dx} = e^{-3x}$.

42. $\frac{dy}{dx} - y\sin x = 0$. Here, P = - sin x. $\text{IF} = e^{\int p\,dx} = e^{\int -\sin x\,dx} = e^{\cos x}$.

43. $xy^2\frac{dy}{dx} = x^3 + y^3$; $\frac{dy}{dx} = \frac{x^3+y^3}{xy^2} = \frac{x^2}{y^2} + \frac{y}{x} = \frac{1}{\left({}^{y}/{}_{x}\right)^2} + \frac{y}{x}$.

44. Homogenous DE.

45. $\frac{y-x}{x} = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} - \frac{y}{x} = -1$. Linear DE.

46. $\frac{dy}{dx} = \frac{y}{x} + \left(\frac{x}{y}\right)^2 \Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{1}{\left({}^{y}/{}_{x}\right)^2}$. Homogenous DE.

47. $\frac{dy}{dx} - \frac{3}{x}y = \left(\frac{y}{x}\right)^2$; $\frac{dy}{dx} - 3\frac{y}{x} = \left(\frac{y}{x}\right)^2$. Homogenous DE.

48. $\frac{dy}{dx} = |x|$. TRY IT YOURSELF.

49. $\frac{dy}{dx} - 4y = 8$. TRY IT YOURSELF.

50. $\sqrt{x}\ \frac{dy}{dx} = x + 3y\ ; \frac{dy}{dx} - \frac{3y}{\sqrt{x}} = x\ ; So, P = -\frac{3}{\sqrt{x}}$

$\text{IF} = e^{\int P\,dx} = e^{\int \frac{-3}{\sqrt{x}}dx} = e^{-6\int \frac{1}{2\sqrt{x}}dx} = e^{-6\sqrt{x}}$

51. $x\frac{dy}{dx} + (5x+2)y = \frac{20}{x}$. $P = 5 + \frac{2}{x}$;

$IF = e^{\int p\,dx} = e^{\int 5+\frac{2}{x}dx} = e^{5x+2logx} = e^{5x}e^{\log x^2} = e^{5x}x^2.$

52. $2\sqrt{x}\ \frac{dy}{dx} + 3xy = 2xe^{-\sqrt{x}}$; Rewriting, $\frac{dy}{dx} + \frac{3}{2}\sqrt{x}\ y = \sqrt{x}\ e^{-\sqrt{x}}$

$\text{P} = \frac{3}{2}\sqrt{x}$. TRY IT YOURSELF.

IX - SOLUTIONS TO CASE STUDIES

I. 1. The circles have their centres at (h , 0) and radii “a”. i.e., k = 0.

Therefore, the equation of the family is $(x - h)^2 + y^2 = a^2$.

2. d) 2, 0.

3. The degree of the differential equation is 1.

4. The solution curve which passes through the point (5,0) and radius 5 is $x^2 + y^2 - 10x = 0$.

5. From solution to Qn.2, yy’ = - (x - h) (OR) y’ = $\frac{-(x-h)}{y}$ = slope of tangent. Therefore, slope of normal = $\frac{y}{x-h}$.

The curve passes thro’ (0,0). Sub x = 0, y = 0, slope of normal = 0.

II. 1. The equation representing this family of parabolas is $y^2 = 4ax$.

2. There is 1 parameter(arbitrary constant). The order is 1.

3. Focus is at 5 units from the origin. a = 5. Equation is $y^2 = 20x$.

4. The y – axis is tangent to all the members of the family and hence

The x – axis is the normal at the origin. Therefore slope is 0.

[OR] Diff, 2yy' = 4a ; $y' = \frac{2a}{y}$ = slope of tangent.

Therefore, slope of normal at (0,0) $= \frac{-1}{slope\ of\ tangent} = \frac{-y}{2a} = 0.$

5. The curve is $y^2 = 20x$. Also, $y' = \frac{10}{y}$ = slope of tangent.

Therefore, $\frac{10}{y} = 1$; y = 10. Sub, $10^2 = 20x \Rightarrow x = 5$. Point is (5,10).

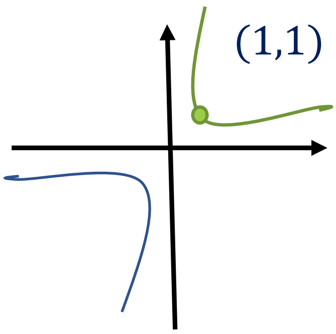
III. 1. The order is 1 , degree is 1.

2. $xy' + y = 0$; $x\frac{dy}{dx} = -y$; $\frac{dy}{y} = -\frac{dx}{x}$; $log\ y = -log\ x + log\ c.$

$\Rightarrow$ log y + log x = log c $\Rightarrow$ xy = c, is the solution.

3. c). The curves are special cases of hyperbolas

called rectangular Hyperbolas.



4. When x = 1, y = 1, we get c = 1.

The solution is xy = 1.

5. The solution curve is strictly decreasing.

IV. 1. a) The equation can be written as $\frac{dy}{dx} = \sin\frac{\pi}{2} = 1$. Degree is 1.

2. c) solving dy/dx = 1, the solution is y = x + c.

A family of parallel lines, the slope of each line being 45°.

3. a) The line passes through (1,1). So, c = 0.

4. a) The y-intercept is – 2. i.e., c = - 2. The equation of the line is

y = x – 2 is satisfied by (3,1).

5. b) The family of lines y = x + c , which have slope 1.

So, the slope of the family of perpendicular lines is – 1.

So, the equation of the family is x + y = c.

V. 1. Let P be the principal deposited with the bank.

The rate of growth is proportional to the principal itself.

i.e., $\frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP$.

2. Seperating the variables, $\frac{dP}{P} = k\,dt$,

Integrating, $\log_e P = kt + \log c$.

$\log_e \frac{P}{c} = kt$; $\frac{P}{c} = e^{kt}$ or $P = ce^{kt}$.

3. When t = 0, P = 1000. When P = 2000, t = ?

Sub, t = 0, P = 1000, we get c = 1000.

Now, $2000 = 1000e^{kt}$, k = 5% = 1/20 ⇒ $2 = e^{\frac{t}{20}}$.

Taking log, $\log 2 = \frac{t}{20}$ ⇒ t = 14, approximately.

4. When t = 10, P = $1000\, e^{\frac{10}{20}} = 1000 \times 1.648 = Rs. 1648$.

5. If r = 10%, P = $1000\, e^{\frac{10}{10}} = 1000 \times 2.718 = Rs. 2718$.

VI. 1. b) 2. c) 3. c) 4. d) 5. b)

END OF CHAPTER 9

// K.MANI , SALEM, TAMILNADU //

X. VECTOR ALGEBRA

VERY SHORT ANSWER TYPE

1. $|\vec{a}| = 5,\ \lambda \in [-4,-2]$,then find the range of values of $|\lambda\vec{a}|$.

2. If $|\vec{a} - \vec{b}| = \sqrt{3}$, where $\vec{a}, \vec{b}$ are unit vectors, find the angle between $\vec{a}$ *and* $\vec{b}$.

3. In the XoY plane, a square of side 1 unit is drawn with one of its sides on positive x-axis. What are the vectors representing the diagonals ?

4. If $\vec{a}$ *and* $\vec{b}$ are collinear vectors acting in the same direction, then find the value of $|\vec{a} + \vec{b}|$.

5. If $\vec{a}$ *and* $\vec{b}$ are collinear vectors acting in opposite directions, then find the value of $|\vec{a} + \vec{b}|$.

6. If $\vec{a}$ *and* $\vec{b}$ are vectors of equal magnitude inclined at an angle $\theta (\neq 0)$ then find the projection of $\vec{a} + \vec{b}$ on $\vec{a} - \vec{b}$.

7. Find the area of the parallelogram whose two adjacent are $2\vec{j}$ *and* $3\vec{k}$.

8. Find the area of the square whose one side is $3\vec{i}$.

9. Evaluate : $\Sigma\, \hat{i} \times \hat{j}$

10. Evaluate : $\Sigma\, \hat{i} \times (\hat{j} \times \hat{k})$

11. Evaluate : $\Sigma\, \hat{i} \times (\hat{j} + \hat{k})$

12. Find the area of the triangle OAB, where O is the origin, A = (5,0,0) and B = (0, -5, 0).

13. Find the unit vector in the direction of the sum of the vectors $2\hat{i} + \hat{j} - 3\hat{k}$ and $\hat{i} + 2\hat{j} + 4\hat{k}$.

14. Find the unit vectors parallel to $3\hat{i} + 2\hat{j} + 6\hat{k}$.

15. Find the unit vectors parallel to the sum of the vectors $\hat{i} - 3\hat{k}$ and $2\hat{i} + 4\hat{j} + 6\hat{k}$.

16. Find a unit vector perpendicular to $2\hat{\imath} + \hat{\jmath} + 3\hat{k}$.

17. Write atleast 3 sets of direction ratios of the vector $3\hat{\imath} - 2\hat{\jmath} - 4\hat{k}$.

18. A cube whose edge length is 5 units has its 3 co-terminous edges along the coordinate axes. Find the position vector of the midpoint of the diagonal through O.

19. The vertices of a triangle are A = (3,1,-2), B = (-2,1,1) and C = (3 -2,-2). What are the direction cosines of the median through A ?

20. The vertices of a triangle are A = (3,1,-2), B = (-2,1,1) and C = (3 -2,-2). Find the position vector of the centroid of ΔABC.

21. If the vectors $-4\hat{\imath} + 6\hat{\jmath} - 8\hat{k}$ and $-6\hat{\imath} + m\hat{\jmath} - 12\hat{k}$ are collinear , then , find the Value of m.

22. Find the vector that is equally inclined to the coordinate axes.

23. If the vectors $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and $\vec{b} = -4\hat{\imath} + m\hat{\jmath} - n\hat{k}$ are such that $\vec{a} = \lambda\vec{b}$, then find the values of λ , m , n.

24. Write a vector perpendicular to $2\hat{\imath} + \hat{\jmath} + 3\hat{k}$.

25. If $\vec{a} = 3\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$, then find $\vec{a}^{\,2}$.

26. Evaluate : $\Sigma\, \hat{\imath} \cdot (\hat{\jmath} + \hat{k})$

27. Evaluate : $\Sigma\, \hat{\imath} \cdot (\hat{\jmath} - \hat{k})$

28. If $\vec{a} = 4\hat{\imath} - 2\hat{\jmath} - 4\hat{k}$, find $\vec{a} \cdot \vec{a}$ and $\vec{a} \cdot (-\vec{a})$.

29. Find the projection of $2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ on x – axis.

30. Find the projection of $2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ on y – axis.

31. Find the projection of $2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ on z – axis.

32. If $\vec{a}$ *and* $\vec{b}$ are vectors with equal magnitudes, what is the angle between $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$?

33. If $|\vec{a}| = 3, |\vec{b}\,| = 4$, find $\vec{a} \cdot \vec{b}$ when $\vec{a}$ *and* $\vec{b}$ are collinear.

34. If $|\vec{a}| = 3, |\vec{b}\,| = 4$, what can be the maximum value of $|\vec{a} + \vec{b}\,|$?

35. If $|\vec{a}| = 3, |\vec{b}| = 4$, what is the minimum value of $|\vec{a} - \vec{b}|$?

36. If $|\vec{a}| = 2$, $\overrightarrow{x} + \overrightarrow{a}$ and $\overrightarrow{x} - \overrightarrow{a}$ are orthogonal, then find $|\overrightarrow{x}|$.

37. Find the projection of $\hat{\imath} + \hat{\jmath}$ on $\hat{\imath} - \hat{\jmath}$.

38. Find the projection of $\hat{k}$ *on* $\hat{\imath} + \hat{\jmath} + \hat{k}$.

39. Find the projection of $\hat{k}$ *on* $\hat{\imath} - \hat{\jmath}$.

40. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors and $|\vec{a}| = 1, |\vec{b}| = 2$, $|\vec{c}| = 3$, then find $(\vec{a} - 2\vec{b} + 3\vec{c}) \cdot (2\vec{a} + \vec{b} - \vec{c})$.

41. If $\vec{a}$ *and* $\vec{b}$ are perpendicular unit vectors then, find $|\vec{a} + \vec{b}|$.

42. If $\vec{a}$ *and* $\vec{b}$ are perpendicular unit vectors then, find $|\vec{a} - \vec{b}|$.

43. If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular unit vectors ,then, find the value of $|\vec{a} + \vec{b} + \vec{c}|$.

44. Find the unit vectors perpendicular to each of the vectors $\overrightarrow{a} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $\overrightarrow{b} = \hat{\imath} + \hat{\jmath} + 3\hat{k}$.

45. ABCD is a rhombus in which $\overrightarrow{AB} = \hat{\imath} + \hat{\jmath}$, $\overrightarrow{BC} = \hat{\jmath} + \hat{k}$. Find $\overrightarrow{AC}$, $\overrightarrow{BD}$.

46. If $\vec{a}$ *and* $\vec{b}$ are two non-zero vectors,then what is the value of $(\overrightarrow{a \times b})^2 + (\overrightarrow{a \cdot b})^2$.

47. Find $\overrightarrow{a} \times \overrightarrow{b}$ where $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{b} = -\hat{\imath} - \hat{\jmath} - \hat{k}$ and comment on the value of $\overrightarrow{a} \times \overrightarrow{b}$.

48. If $\overrightarrow{a} = \hat{\imath} + \hat{\jmath}$, $\overrightarrow{b} = \hat{\jmath} + \hat{k}$, $\overrightarrow{c} = \hat{k} + \hat{\imath}$, find $\overrightarrow{a} \cdot (\overrightarrow{b} \times \overrightarrow{c})$.

49. If $3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$ and $2\hat{\imath} + a\hat{\jmath} + 6\hat{k}$ are collinear, then what is a?

50. Evaluate : $\Sigma \overrightarrow{a} \times (\overrightarrow{b} + \overrightarrow{c})$

51. Evaluate : $\Sigma \hat{\imath} \times (\hat{\jmath} - \hat{k})$

52. Find vectors of magnitudes 5 units parallel to $3\hat{\imath} - 2\hat{\jmath} + 6\hat{k}$.

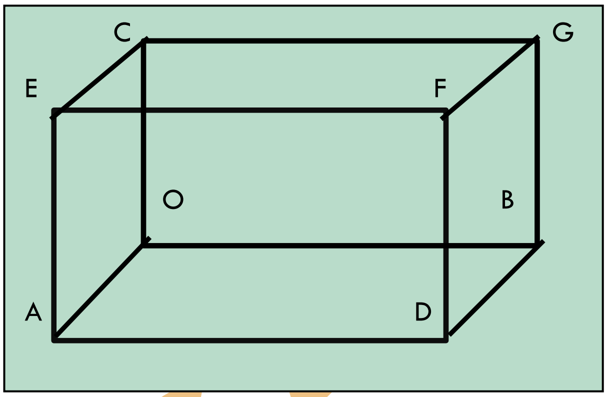
53. If θ is the angle between the unit vectors $\hat{a}$ *and* $\hat{b}$, prove that $sin\frac{\theta}{2} = \frac{1}{2}|\hat{a} - \hat{b}|$.

54. If θ is the angle between the unit vectors $\hat{a}$ *and* $\hat{b}$, prove that

$cos\frac{\theta}{2} = \frac{1}{2}|\hat{a} + \hat{b}|$.

55. P,Q are the points of trisection of the line segment AB, where

$\overrightarrow{OA} = \hat{\imath}$, $\overrightarrow{OB} = \hat{\jmath}$. Find $\overrightarrow{PQ}$.



56. In the adjoining diagram,

the lengths of the edges OA,OB,OC of the cuboid are 3,4,5. Find the angle between the diagonals OF and CD.

57. Find a vector of magnitude 5 units in the direction opposite to $3\hat{\imath} - 6\hat{\jmath} - 2\hat{k}$.

58. The position vectors of the vertices A , B and C of a triangle are $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ respectively. Find the length of the median through A.

59. If the points P, Q and R with position vectors $2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}$, $\hat{\imath} + 2\hat{\jmath} + 7\hat{k}$, $3\hat{\imath} + m\hat{\jmath} - \hat{k}$ are collinear, find m.

60. Prove that the points P , Q and R with position vectors $\hat{\imath} + \hat{\jmath}$, $\hat{\jmath} + \hat{k}$, $\hat{k} + \hat{\imath}$ form a triangle.

61. Prove that the points P , Q and R with position vectors $\hat{\imath} + \hat{\jmath}$, $\hat{\jmath} + \hat{k}$, $\hat{k} + \hat{\imath}$ are non-collinear.

X.VECTOR ALGEBRA

CASE STUDY BASED MCQS

I. Given three vectors $\vec{a} = 2\hat{\imath} - \hat{\jmath} + \hat{k}$, $\vec{b} = \hat{\imath} - 3\hat{\jmath} - 5\hat{k}$, $\vec{c} = 3\hat{\imath} - 4\hat{\jmath} - 4\hat{k}$. Now, answer the questions that follow.

1. The triangle formed by these vectors is

   a) equilateral b) isosceles c) right angled d) rt.angled isosceles.

2. The area of the triangle is

   a) $\sqrt{210}$ b) 210 c) $\frac{1}{2}\sqrt{210}$ d) $2\sqrt{210}$.

3. If $|\vec{a} + \vec{b}| = |\lambda \vec{c}|$, then $\lambda =$

   a) 1 b) – 1 c) ± 1 d) can be any real number.

4. The projection of $\vec{a}$ on $\vec{b}$ is

   a) $\sqrt{35}$ b) 0 c) $\frac{35}{\sqrt{41}}$ d) $\frac{6}{\sqrt{41}}$

5. The angles of the triangle are

   a) $90°, 30°, 60°$ b) $90°, \cos^{-1}\sqrt{\frac{35}{41}}, \cos^{-1}\sqrt{\frac{6}{41}}$

   c) $60°, 60°, 60°$ d) $90°, \cos^{-1}\frac{35}{\sqrt{41}}, \cos^{-1}\frac{6}{\sqrt{41}}$

II. Let A , B , C be three points with position vectors $\hat{\imath} + \hat{\jmath}$, $\hat{\jmath} + \hat{k}$, $\hat{k} + \hat{\imath}$.

Now, answer the questions that follow.

1. The three points

   a) form an equilateral Δ b) form an isosceles Δ

   c) form a right angled Δ d) are collinear.

2. The lengths of the sides of the triangle are

   a) 2 , 2 , 2 b) 1 , 2 , 5 c) $\sqrt{2}$, $\sqrt{2}$, $\sqrt{2}$ d) 2, $\sqrt{2}$, $\sqrt{6}$.

3. The area of the triangle is

a) $\sqrt{3}$ b) $\sqrt{3}/4$ c) $\sqrt{3}/2$ d) $2\sqrt{3}$.

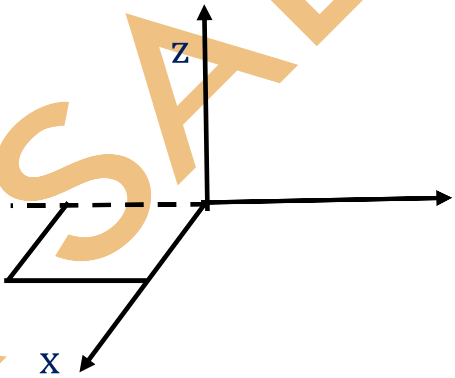
4. The measure of the angle $\angle ABC$ =

a) 30° b) 45° c) 60° d) 90° .

5. The area of the parallelogram with AB,AC as adjacent sides is

a) $\sqrt{2}$ b) $\sqrt{6}$ c) $\sqrt{3}$ d) $\sqrt{3}/2$.

III. The adjacent sides AB and AD of a parallelogram ABCD, are given by $\overrightarrow{AB} = \vec{a} = 3\hat{\imath}$, $\overrightarrow{AD} = \vec{b} = -4\,\hat{\jmath}$. Now, answer the following questions.



1. The area of the parallelogram ABCD is

a) 12 b) 6 c) 18 d) 24.

2. The area of the parallelogram with $\overrightarrow{AC}$ and $\overrightarrow{BD}$ as adjacent sides is

a) 6 b) 24 c) 48 d) 12.

3. The ratio of the areas mentioned in Qn. 1 and Qn. 2 is

a) 1:2 b) 1:3 c) 2:3 d) 3:4.

4. $(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) =$

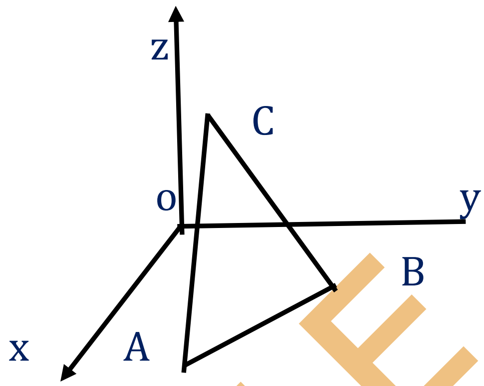
a) $-24\,\hat{k}$ b) $24\,\hat{\imath}$ c) $\vec{0}$ d) $24\,\hat{k}$.

5. The unit vectors perpendicular to $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ are

a) $\pm\,\hat{\imath}$ b) $\pm\,\hat{\jmath}$ c) $\pm\,\hat{k}$ d) $\hat{\imath} \pm \hat{\jmath}$.

IV. A,B and C are the corner points of a thin triangular plate of uniform thickness and constant density.

A = (4,2,0), B = (1,3,0), C = (1,1,3).



Now, answer the following questions.

1. The vector from C to the midpoint P of side AB is

a) $\frac{3}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath} + 3\hat{k}$ b) $-\frac{3}{2}\hat{\imath} - \frac{3}{2}\hat{\jmath} + 3\hat{k}$

c) $\frac{3}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath} - 3\hat{k}$ d) $\frac{2}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} - 3\hat{k}$

2. If point G_1 divides CP in the ratio 2 : 1, then position vector of G_1 is

a) $2\hat{\imath} + 2\hat{\jmath} + \hat{k}$ b) $\frac{7}{2}\hat{\imath} + \frac{7}{2}\hat{\jmath} + \frac{3}{2}\hat{k}$

c) $2\hat{\imath} + 3\hat{\jmath} + \frac{3}{2}\hat{k}$ d) $\frac{3}{4}\hat{\imath} + \frac{3}{4}\hat{\jmath} + 2\hat{k}$

3. The vector from A to the midpoint Q of side BC is

a) $3\hat{\imath} - \frac{3}{2}\hat{k}$ b) $\frac{5}{2}\hat{\imath} + 2\hat{\jmath} + \frac{3}{4}\hat{k}$

c) $-3\hat{\imath} + \frac{3}{2}\hat{k}$ d) $3\hat{\imath} + \frac{3}{2}\hat{k}$

4. If the point G_2 divides AQ in the ratio 2 : 1, then position vector of G_2 is

a) $3\hat{\imath} + 2\hat{\jmath} + \frac{1}{2}\hat{k}$ b) $\frac{9}{2}\hat{\imath} + 3\hat{\jmath} + \frac{3}{4}\hat{k}$

c) $2\hat{\imath} + 2\hat{\jmath} + \hat{k}$ d) $\frac{5}{2}\hat{\imath} + 2\hat{\jmath} + \frac{3}{4}\hat{k}$

5. If point G_3 divides BR in the ratio 2 : 1, then position vector of G_3 is

a) $\frac{5}{4}\hat{\imath} + 2\hat{\jmath} + \frac{1}{2}\hat{k}$ b) $2\hat{\imath} + 2\hat{\jmath} + \hat{k}$

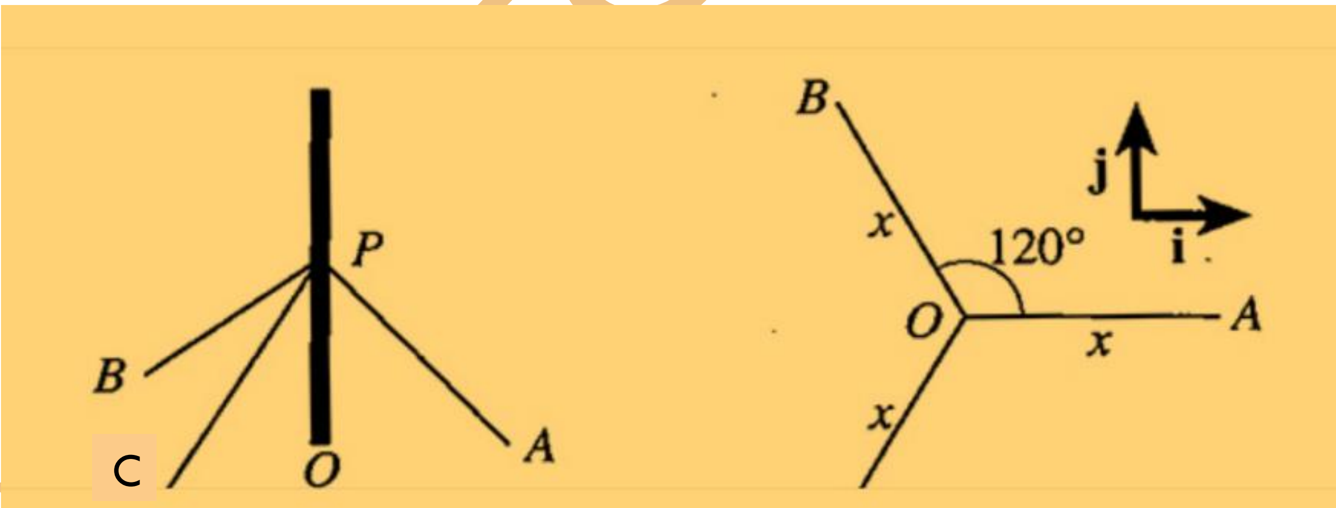
c) $3\hat{\imath} + 2\hat{\jmath} + \frac{3}{4}\hat{k}$ d) $\frac{5}{2}\hat{\imath} + 2\hat{\jmath} + \frac{3}{4}\hat{k}$

*[NOTE : THIS PROBLEM DISCUSSED THE PROCEDURE FOR FINDING THE SPECIAL POINT IN A TRIANGLE CALLED THE CENTROID. IN PHYSICS, THIS POINT IS CALLED **THE CENTRE OF MASS**].*

V. A vertical aerial is supported by three straight cables, each attached to the aerial at a point P, 30 metres up the aerial. The cables are attached to the horizontal ground at points A,B and C , each x metres from the foot O of the aerial, and situated symmetrically around it(at equal distance around the aerial). See the diagram.

Suppose that $\hat{\imath}$ is the unit vector in the direction $\overrightarrow{OA}$, $\hat{\jmath}$ is the unit vector perpendicular to $\hat{\imath}$ in the plane of the ground, as shown in the Plan view, and $\hat{k}$ is the unit vector in the direction $\overrightarrow{OP}$.

Now, answer the following questions :



1. The expression for $\overrightarrow{OA}$ is

a) $-x\hat{\imath}$ b) $x\hat{\imath} + x\hat{\jmath}$ c) $x\hat{\imath}$ d) $x\hat{\imath} - x\hat{\jmath}$.

2. The expression for $\overrightarrow{OB}$ is

a) $-\frac{1}{2}x\hat{\imath} - \frac{\sqrt{3}}{2}x\hat{\jmath}$

b) $\frac{1}{2}x\hat{\imath} - \frac{\sqrt{3}}{2}x\hat{\jmath}$

c) $-\frac{1}{2}x\hat{\imath} + \frac{\sqrt{3}}{2}x\hat{\jmath}$

d) $\frac{1}{2}x\hat{\imath} + \frac{\sqrt{3}}{2}x\hat{\jmath}$

3. The expression for $\overrightarrow{OC}$ is

a) $-\frac{1}{2}x\hat{\imath} - \frac{\sqrt{3}}{2}x\hat{\jmath}$

b) $\frac{1}{2}x\hat{\imath} - \frac{\sqrt{3}}{2}x\hat{\jmath}$

c) $-\frac{1}{2}x\hat{\imath} + \frac{\sqrt{3}}{2}x\hat{\jmath}$

d) $\frac{1}{2}x\hat{\imath} + \frac{\sqrt{3}}{2}x\hat{\jmath}$

4. The vector $\overrightarrow{AP}$ is

a) $-x\hat{\imath} - 30\hat{k}$ b) $x\hat{\imath} - 30\hat{k}$ c) $-x\hat{\imath} + 30\hat{k}$ d) $x\hat{\imath} + 30\hat{k}$.

5. The vector $\overrightarrow{BP}$ is

a) $-\frac{1}{2}x\hat{\imath} - \frac{\sqrt{3}}{2}x\hat{\jmath} + 30\hat{k}$

b) $\frac{1}{2}x\hat{\imath} - \frac{\sqrt{3}}{2}x\hat{\jmath} + 30\hat{k}$

c) $-\frac{1}{2}x\hat{\imath} + \frac{\sqrt{3}}{2}x\hat{\jmath} + 30\hat{k}$

d) $\frac{1}{2}x\hat{\imath} + \frac{\sqrt{3}}{2}x\hat{\jmath} + 30\hat{k}$.

X. VECTOR ALGEBRA – SOLUTIONS

1. $|\vec{a}| = 5. |\lambda\vec{a}| = |\lambda||\vec{a}| = |-4|\cdot 5 = 20, when\ \lambda = -4,$

   and $|\lambda\vec{a}| = |\lambda||\vec{a}| = |-2|\cdot 5 = 10, when\ \lambda = -2.$
   Therefore, $|\lambda\vec{a}| \in [10, 20].$

2. $|\vec{a} - \vec{b}| = \sqrt{3}$, where $\vec{a}, \vec{b}$ are unit vectors. i.e., $|\vec{a}| = |\vec{b}| = 1.$

   $|\vec{a} - \vec{b}| = \sqrt{3} \Rightarrow \vec{a}^2 + \vec{b}^2 - 2\vec{a}\cdot\vec{b} = 3$

   $\Rightarrow -2\vec{a}\cdot\vec{b} = 1, \quad \vec{a}^2 = |\vec{a}|^2 = 1$

   $\Rightarrow -2ab\cos\theta = 1 \Rightarrow \theta = 120°.$

3. $\overrightarrow{OB}$ *and* $\overrightarrow{AC}$ are the diagonals. $\overrightarrow{OA} = \vec{\imath}, \ \overrightarrow{OC} = \vec{\jmath}.$

   $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = \hat{\imath} + \hat{\jmath}; \ \overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = -\hat{\imath} + \hat{\jmath}.$

4. ***DO IT YOURSELF*** [ANS : $|\vec{a}| + |\vec{b}|$].

5. ***DO IT YOURSELF*** $|\vec{a}| - |\vec{b}|\ OR\ |\vec{b}| - |\vec{a}|$

6. Projection of $\vec{a} + \vec{b}$ *on* $\vec{a} - \vec{b} = \dfrac{(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})}{|\vec{a}-\vec{b}|\,|\vec{a}-\vec{b}|}$

   $= \dfrac{\vec{a}^2 - \vec{b}^2}{|\vec{a}-\vec{b}|\,|\vec{a}-\vec{b}|} = 0$, since $|\vec{a}| = |\vec{b}|.$

7. Area of the parallelogram $= |2\hat{\jmath} \times 3\hat{k}| = |6\hat{\imath}| = 6\ sq.units.$

8. The one side is $3\hat{\imath}$. So, the other side is $3\hat{\jmath}$.

   Area $= |3\hat{\imath} \times 3\hat{\jmath}| = |9\hat{k}| = 9\ sq.units.$

   * [The other side can also be taken as $-3\hat{\jmath}\ or\ 3\hat{k}\ or\ -3\hat{k}$].

9. $\Sigma\,\hat{\imath} \times \hat{\jmath} = \hat{\imath} \times \hat{\jmath} + \hat{\jmath} \times \hat{k} + \hat{k} \times \hat{\imath} = \hat{\imath} + \hat{\jmath} + \hat{k}.$

10. $\Sigma\,\hat{\imath} \times (\hat{\jmath} \times \hat{k}) = \hat{\imath} \times (\hat{\jmath} \times \hat{k}) + \hat{\jmath} \times (\hat{k} \times \hat{\imath}) + \hat{k} \times (\hat{\imath} \times \hat{\jmath})$

    $= \hat{\imath} \times \hat{\imath} + \hat{\jmath} \times \hat{\jmath} + \hat{k} \times \hat{k} = \vec{0}.$

11. $\Sigma\,\hat{\imath} \times (\hat{\jmath} + \hat{k}) = \hat{\imath} \times (\hat{\jmath} + \hat{k}) + \hat{\jmath} \times (\hat{k} + \hat{\imath}) + \hat{k} \times (\hat{\imath} + \hat{\jmath})$

    $= \hat{\imath} \times \hat{\jmath} + \hat{\imath} \times \hat{k} + \hat{\jmath} \times \hat{k} + \hat{\jmath} \times \hat{\imath} + \hat{k} \times \hat{\imath} + \hat{k} \times \hat{\jmath}$

    $= \vec{0}$, since $\hat{\imath} \times \hat{\jmath} = -\hat{\jmath} \times \hat{\imath}, etc.,$

12. A = (5,0,0) , B = (0, −5, 0). Therefore, $\overrightarrow{OA} = 5\vec{\imath}, \overrightarrow{OB} = -5\vec{\jmath}.$

    Area of $\Delta OAB = \frac{1}{2}|\overrightarrow{OA} \times \overrightarrow{OB}| = \frac{1}{2}|5\vec{\imath} \times -5\vec{\jmath}| = \frac{25}{2}.$

13. Let $\vec{a} = 2\hat{\imath} + \hat{\jmath} - 3\hat{k}, \ \vec{b} = \hat{\imath} + 2\hat{\jmath} + 4\hat{k}.$

Let $\overrightarrow{a} + \overrightarrow{b} = 3\hat{\imath} + 3\hat{\jmath} + \hat{k} = \overrightarrow{c}, say.$ $|\overrightarrow{c}| = \sqrt{19}.$

So, the unit vector in the direction of $\overrightarrow{c} = \hat{c} = \frac{\overrightarrow{c}}{|\overrightarrow{c}|} = \frac{3\hat{\imath}+3\hat{\jmath}+\hat{k}}{\sqrt{19}}.$

14. Let $\overrightarrow{a} = 3\hat{\imath} + 2\hat{\jmath} + 6\hat{k}$. $|\overrightarrow{a}| = 7.$

Therefore, the unit vectors parallel to $\overrightarrow{a}$ are $\pm\hat{a} = \pm\frac{3\hat{\imath}+2\hat{\jmath}+6\hat{k}}{7}.$

15. Let $\overrightarrow{a} = \hat{\imath} - 3\hat{k}$, $\overrightarrow{b} = 2\hat{\imath} + 4\hat{\jmath} + 6\hat{k}.$

Let $\overrightarrow{a} + \overrightarrow{b} = 3\hat{\imath} + 4\hat{\jmath} + 3\hat{k} = \overrightarrow{c}, say.$ $|\overrightarrow{c}| = \sqrt{34}.$

So, the unit vectors parallel to $\overrightarrow{c} = \pm\hat{c} = \pm\frac{\overrightarrow{c}}{|\overrightarrow{c}|} = \pm\frac{3\hat{\imath}+3\hat{\jmath}+\hat{k}}{\sqrt{19}}.$

16. Let $\overrightarrow{a} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$. A vector perpendicular to $\overrightarrow{a}$ can be

$\hat{\imath} + \hat{\jmath} - \hat{k} = \overrightarrow{c}$, say. Therefore, $\pm\hat{c} = \pm\frac{\overrightarrow{c}}{|\overrightarrow{c}|} = \pm\frac{\hat{\imath}+\hat{\jmath}-\hat{k}}{\sqrt{3}}.$

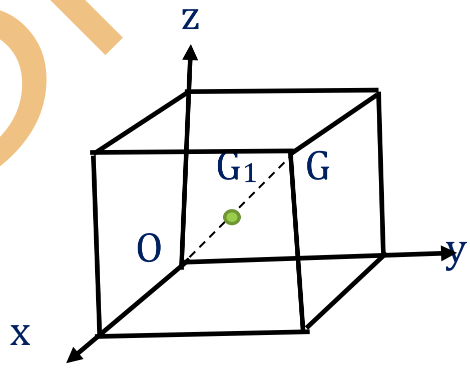
17. Any three sets of numbers proportional to $(3, -2, -4)$.

18. G = (5,5,5).

$\overrightarrow{OG} = 5\hat{\imath} + 5\hat{\jmath} + 5\hat{k}$.

Let G_1 be the midpoint of OG.

$\overrightarrow{OG_1} = \frac{5}{2}\hat{\imath} + \frac{5}{2}\hat{\jmath} + \frac{5}{2}\hat{k}$



19. ***DO IT YOURSELF.*** [ANS : $\frac{-5}{\sqrt{43}}, \frac{-3}{\sqrt{43}}, \frac{3}{\sqrt{43}}$]

20. Given, A = (3,1,-2), B = (-2,1,1) and C = (3 -2,-2).

The position vector of the centroid $= \frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3} = \frac{\frac{4}{3}\hat{\imath} - \hat{k}}{3}.$

21. Let $\overrightarrow{a} = -4\hat{\imath} + 6\hat{\jmath} - 8\hat{k}$ and $\overrightarrow{b} = -6\hat{\imath} + m\hat{\jmath} - 12\hat{k}.$

Since $\overrightarrow{a}$, $\overrightarrow{b}$ are collinear, $\overrightarrow{b} = \lambda\overrightarrow{a}$.

Therefore, $-6\hat{\imath} + m\hat{\jmath} - 12\hat{k} = \lambda(-4\hat{\imath} + 6\hat{\jmath} - 8\hat{k})$; $-4\lambda = -6;$

$\lambda = \frac{3}{2}$; $m = 9.$

22. ***DO IT YOURSELF***

23. $\vec{a} = 2\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$ and $\vec{b} = -4\hat{\imath} + m\hat{\jmath} - n\hat{k}$ are such that $\vec{a} = \lambda\vec{b}$.

So, $2\hat{\imath} - 3\hat{\jmath} + 4\hat{k} = -4\lambda\hat{\imath} + m\lambda\hat{\jmath} - n\lambda\hat{k}$; $\lambda = \frac{-1}{2}$; $m = 6$; $n = 8$.

24. Let $\vec{a} = 2\hat{\imath} + \hat{\jmath} + 3\hat{k}$. The vector, $\hat{\imath} + 4\hat{\jmath} - 2\hat{k}$ is $\perp \vec{a}$.

25. $\vec{a} = 3\hat{\imath} - 2\hat{\jmath} - 6\hat{k}$, then $\vec{a}^{\,2} = \vec{a} \cdot \vec{a} = |\vec{a}|^2 = a^2$. $\vec{a}^{\,2} = 49$.

26. $\Sigma\, \hat{\imath} \cdot (\hat{\jmath} + \hat{k}) = \hat{\imath} \cdot (\hat{\jmath} + \hat{k}) + \hat{\jmath} \cdot (\hat{k} + \hat{\imath}) + \hat{k} \cdot (\hat{\imath} + \hat{\jmath})$

$= \hat{\imath} \cdot \hat{\jmath} + \hat{\imath} \cdot \hat{k} + \hat{\jmath} \cdot \hat{k} + \hat{\jmath} \cdot \hat{\imath} + \hat{k} \cdot \hat{\imath} + \hat{k} \cdot \hat{\jmath}$

$= 0 + 0 + 0 + 0 + 0 + 0 = 0.$

27. $\Sigma\, \hat{\imath} \cdot (\hat{\jmath} - \hat{k}) = 0$, Similar to Qn. 26.

28. If $\vec{a} = 4\hat{\imath} - 2\hat{\jmath} - 4\hat{k}$, find $\vec{a} \cdot \vec{a}$ and $\vec{a} \cdot (-\vec{a})$.

$\vec{a} \cdot \vec{a} = 36$; $\vec{a} \cdot (-\vec{a}) = -\vec{a} \cdot \vec{a} = -36$.

29. projection of $2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ on x – axis. x-axis can be taken as $\hat{\imath}$.

projection of $2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$ on x – axis $= \frac{(2\hat{\imath}+3\hat{\jmath}+4\hat{k}) \cdot \hat{\imath}}{|\hat{\imath}|} = \frac{2}{1} = 2.$

30. y-axis can be taken as $\hat{\jmath}$. Proceed as in Qn.29.

31. z-axis can be taken as $\hat{k}$. Proceed as in Qn.29.

32. ***DO IT YOURSELF.*** [ANS : $\pi/2$]

33. ***DO IT YOURSELF.*** [ANS : 12 or – 12].

34. ***DO IT YOURSELF.*** [ANS : 7]

35. ***DO IT YOURSELF.*** [ANS : 1]

36. $|\vec{a}| = 2$, $\vec{x} + \vec{a}$ and $\vec{x} - \vec{a}$ are orthogonal.

$\therefore (\vec{x} + \vec{a}) \cdot (\vec{x} - \vec{a}) = 0 \Rightarrow \vec{x}^{\,2} - \vec{a}^{\,2} = 0$

$\Rightarrow \vec{x}^{\,2} - 4 = 0$; $\therefore |\vec{x}| = 2.$

37. The projection of $\hat{\imath} + \hat{\jmath}$ on $\hat{\imath} - \hat{\jmath} = \frac{(\hat{\imath}+\hat{\jmath}) \cdot (\hat{\imath}-\hat{\jmath})}{|\hat{\imath}-\hat{\jmath}|} = \frac{0}{\sqrt{2}} = 0.$

38. The projection of $\hat{k}$ *on* $\hat{\imath} + \hat{\jmath} + \hat{k} = \frac{(\hat{k}) \cdot (\hat{\imath}+\hat{\jmath}+\hat{k})}{|\hat{\imath}+\hat{\jmath}+\hat{k}|} = \frac{1}{\sqrt{3}}$.

39. The projection of $\hat{k}$ *on* $\hat{\imath} - \hat{\jmath} = \frac{(\hat{k}) \cdot (\hat{\imath}-\hat{\jmath})}{|\hat{\imath}-\hat{\jmath}|} = \frac{0}{\sqrt{2}} = 0.$

40. $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually $\perp$ vectors. $\therefore \overrightarrow{a}\cdot\overrightarrow{b} = \overrightarrow{b}\cdot\overrightarrow{c} = \overrightarrow{c}\cdot\overrightarrow{a} = 0.$

$|\vec{a}| = 1, |\vec{b}| = 2, |\vec{c}| = 3.$

$(\vec{a} - 2\vec{b} + 3\vec{c}) \cdot (2\vec{a} + \vec{b} - \vec{c}) = 2\overrightarrow{a}^2 - 2\overrightarrow{b}^2 - 3\overrightarrow{c}^2, \overrightarrow{a}\cdot\overrightarrow{b} = 0$, etc.,

$= 2.(1^2) - 2.(2^2) - 3.(3^2) = -33.$

41. $\vec{a} \perp \vec{b}. \therefore \overrightarrow{a}\cdot\overrightarrow{b} = 0.$ Also, $|\vec{a}| = 1, |\vec{b}| = 1.$

Now, $|\vec{a} + \vec{b}|^2 = (\overrightarrow{a} + \vec{b})^2 = \vec{a}^2 + \vec{b}^2 + 2\vec{a}\cdot\vec{b}$

$= 1^2 + 1^2 + 0, \ \overrightarrow{a}^2 = |\overrightarrow{a}|^2$

$= 2. \quad \therefore |\vec{a} + \vec{b}| = 1.$

42. $\vec{a} \perp \vec{b}. \therefore \overrightarrow{a}\cdot\overrightarrow{b} = 0.$ Also, $|\vec{a}| = 1, |\vec{b}| = 1.$

Now, $|\vec{a} - \vec{b}|^2 = (\overrightarrow{a} - \vec{b})^2 = \vec{a}^2 + \vec{b}^2 - 2\vec{a}\cdot\vec{b}$

$= 1^2 + 1^2 - 0$, since $\overrightarrow{a}^2 = |\overrightarrow{a}|^2$

$= 2. \quad \therefore |\vec{a} - \vec{b}| = 1.$

43. $\vec{a}, \vec{b}$ and $\vec{c}$ are mutually perpendicular unit vectors.

$\therefore \overrightarrow{a}\cdot\overrightarrow{b} = \overrightarrow{b}\cdot\overrightarrow{c} = \overrightarrow{c}\cdot\overrightarrow{a} = 0. |\overrightarrow{a}| = |\overrightarrow{b}| = |\overrightarrow{c}| = 1.$

Now, $|\vec{a} + \vec{b} + \vec{c}|^2 = \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2\vec{a}\cdot\vec{b} + 2\overrightarrow{b}\cdot\overrightarrow{c} + 2\overrightarrow{c}\cdot\overrightarrow{a}$

$= 1^2 + 1^2 + 1^2 + 0 + 0 + 0 = 3.$

$\therefore |\vec{a} + \vec{b} + \vec{c}| = 1.$

44. $\overrightarrow{a} = 3\hat{\imath} - \hat{\jmath} + 2\hat{k}, \ \overrightarrow{b} = \hat{\imath} + \hat{\jmath} + 3\hat{k}.$

The unit vectors perpendicular to

each of the vectors are $\pm\hat{n} = \pm \frac{\overrightarrow{a}\times\overrightarrow{b}}{|\overrightarrow{a}\times\overrightarrow{b}|}.$

Now, $\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 2 \\ 1 & 1 & 3 \end{vmatrix} = -5\hat{\imath} - 7\hat{\jmath} + 4\hat{k}.$

$|\overrightarrow{a} \times \overrightarrow{b}| = \sqrt{90}. \quad \therefore \pm\hat{n} = \pm \frac{-5\hat{\imath} - 7\hat{\jmath} + 4\hat{k}}{\sqrt{90}}.$

45. ***DO IT YOURSELF.*** [ANS : $\hat{\imath} + 2\hat{\jmath} + \hat{k}, \ -\hat{\imath} + \hat{k}$].

46. $(\vec{a} \times \vec{b})^2 + (\vec{a} \cdot \vec{b})^2 = (ab\sin\theta)^2 + (ab\cos\theta)^2$

$$= a^2b^2\sin^2\theta + a^2b^2\cos^2\theta$$

$$= a^2b^2 = |\vec{a}|^2|\vec{b}|^2.$$

47. $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$ and $\vec{b} = -\hat{\imath} - \hat{\jmath} - \hat{k}$.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{vmatrix} = \vec{0}.$$

Note that, $\vec{b} = -\vec{a}$. that is ., $\vec{a}$, $\vec{b}$ are collinear.

48. If $\vec{a} = \hat{\imath} + \hat{\jmath}$, $\vec{b} = \hat{\jmath} + \hat{k}$, $\vec{c} = \hat{k} + \hat{\imath}$.

$(\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{vmatrix} = \hat{\imath} + \hat{\jmath} - \hat{k}$. Now,find $\vec{a} \cdot (\vec{b} \times \vec{c})$.

49. $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$ and $\vec{b} = 2\hat{\imath} + a\hat{\jmath} + 6\hat{k}$ are colloinear.

i.e., $\vec{a} = \lambda\vec{b}$, for some *real number* λ.

Therefore, $3\hat{\imath} + 2\hat{\jmath} + 9\hat{k} = 2\lambda\hat{\imath} + a\lambda\hat{\jmath} + 6\lambda\hat{k}$.

Note that $\lambda = \frac{3}{2}$. Hence, $a\lambda = 2$ gives a $= \frac{4}{3}$.

50. $\Sigma\, \vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

$$= \vec{a} \times \vec{b} + \vec{a} \times \vec{c} + \vec{b} \times \vec{c} + \vec{b} \times \vec{a} +$$

$$\vec{c} \times \vec{a} + \vec{c} \times \vec{b} = \vec{0},\ \because \vec{a} \times \vec{b} = -\vec{b} \times \vec{a}.$$

51. $\Sigma\, \hat{\imath} \times (\hat{\jmath} - \hat{k}) = \hat{\imath} \times (\hat{\jmath} - \hat{k}) + \hat{\jmath} \times (\hat{k} - \hat{\imath}) + \hat{k} \times (\hat{\imath} - \hat{\jmath})$

$$= \hat{\imath} \times \hat{\jmath} - \hat{\imath} \times \hat{k} + \hat{\jmath} \times \hat{k} - \hat{\jmath} \times \hat{\imath} +$$

$$\hat{k} \times \hat{\imath} - \hat{k} \times \hat{\jmath} = 2(\hat{\imath} + \hat{\jmath} + \hat{k}).$$

[Use , $\hat{\imath} \times \hat{\jmath} = \hat{k}$ *etc., and* $\hat{\imath} \times \hat{\jmath} = -\hat{\jmath} \times \hat{\imath}$ *etc.,*

52. Let $\vec{a} = 3\hat{\imath} - 2\hat{\jmath} + 6\hat{k}$; $|\vec{a}| = 7$

the unit vectors parallel to $\vec{a}$ are $\pm\hat{a} = \pm\frac{3\hat{\imath} - 2\hat{\jmath} + 6\hat{k}}{7}$.

vectors of magnitudes 5 units parallel to $3\hat{\imath} - 2\hat{\jmath} + 6\hat{k}$ are

$\pm 5\hat{a} = \pm 5 \frac{3\hat{\imath} - 2\hat{\jmath} + 6\hat{k}}{7}$.

53. θ is the angle between the unit vectors $\hat{a}$ *and* $\hat{b}$.

Consider, $|\vec{a} - \vec{b}|^2 = (\overrightarrow{a} - \vec{b})^2 = \vec{a}^2 + \vec{b}^2 - 2\vec{a}\cdot\vec{b}$

$= 1^2 + 1^2 - 2.1.1 \cos\theta$, since $\overrightarrow{a}^2 = |\overrightarrow{a}|^2$

$= 2 - 2\cos\theta$

$= 2.\, 2\sin^2\frac{\theta}{2} \Rightarrow |\vec{a} - \vec{b}| = 2\sin\frac{\theta}{2}$.

54. Proceed as in Qn. 53, taking $|\vec{a} + \vec{b}|^2$.

55. P,Q are the points of trisection of the line segment AB, where

$\overrightarrow{OA} = \hat{\imath}, \overrightarrow{OB} = \hat{\jmath}$. AP : PB = 1:2 and AQ : QB = 2:1

$\overrightarrow{OP} = \frac{\hat{\jmath} + 2\hat{\imath}}{3}$; $\overrightarrow{OQ} = \frac{2\hat{\jmath} + \hat{\imath}}{3}$. Now, $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \frac{-\hat{\imath} + \hat{\jmath}}{3}$.

56. The diagonals are perpendicular.

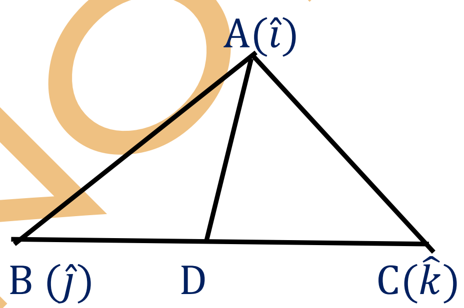
57. Let $\overrightarrow{a} = 3\hat{\imath} - 6\hat{\jmath} - 2\hat{k}$. The vector in the direction opposite to $\overrightarrow{a}$

Is $-\overrightarrow{a} = -3\hat{\imath} + 6\hat{\jmath} + 2\hat{k} = \vec{b}$, say.

Therefore, a vector of magnitude 5 units in the direction opposite

to $\overrightarrow{a}$ is $5\hat{b} = 5\frac{\vec{b}}{|\vec{b}|} = 5\frac{-3\hat{\imath} + 6\hat{\jmath} + 2\hat{k}}{7} = \frac{5}{7}(-3\hat{\imath} + 6\hat{\jmath} + 2\hat{k})$.

58.



$\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA}$

$= \frac{\hat{\jmath} + \hat{k}}{2} - \hat{\imath} = \frac{-2\hat{\imath} + \hat{\jmath} + \hat{k}}{2}$

The length of the median AD $= |\overrightarrow{AD}| = \frac{\sqrt{6}}{2} = \sqrt{\frac{3}{2}}$.

59. ***DO IT YOURSELF.*** [ANS : m = 10].

60. ***DO IT YOURSELF. [ANS:*** $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$]

61. ***DO IT YOURSELF.*** [ANS : No two of $\overrightarrow{PQ}, \overrightarrow{QR}, \overrightarrow{RP}$ *are parallel*.

Also, $\overrightarrow{PQ} + \overrightarrow{QR} + \overrightarrow{RP} = 0$.]

CHAPTER X – SOLUTIONS TO CASE STUDIES

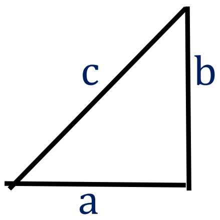
I. 1. Note that $\overrightarrow{a} + \overrightarrow{b} = \overrightarrow{c}$. *Therefore, a triangle is formed.*

Also, $\overrightarrow{a}$ *is perpendicular to* $\overrightarrow{b}$. The Δ is right angled.

Let α, β *and* γ be the angles.

$\alpha =$ Angle between $\overrightarrow{a}$ *and* $\overrightarrow{b} = 90°$.

$\beta =$ Angle between $\overrightarrow{b}$ *and* $\overrightarrow{c}$; $\gamma =$ Angle between $\overrightarrow{c}$ *and* $\overrightarrow{a}$



$$\beta = \cos^{-1}\left\{\frac{(1,-3,-5)\cdot(3,-4,-4)}{\sqrt{35}\,\sqrt{41}}\right\} = \cos^{-1}\sqrt{\frac{35}{41}}$$

$$\gamma = \cos^{-1}\left\{\frac{(2,-1,1)\cdot(3,-4,-4)}{\sqrt{6}\,\sqrt{41}}\right\} = \cos^{-1}\sqrt{\frac{6}{41}}$$

2. $a = \sqrt{6}$; $b = \sqrt{35}$. $\Delta = \frac{1}{2}ab = \frac{1}{2}\sqrt{210}$.

3. c) ± 1

4. Since, $\overrightarrow{a}$ *is perpendicular to* $\overrightarrow{b}$, projection is 0.

5. b)

II. 1. $\overrightarrow{OA} = \hat{\imath} + \hat{\jmath}$, $\overrightarrow{OB} = \hat{\jmath} + \hat{k}$ $\overrightarrow{OC} = \hat{k} + \hat{\imath}$.

$\overrightarrow{AB} = \hat{k} - \hat{\imath}$, $\overrightarrow{BC} = \hat{\imath} - \hat{\jmath}$, $\overrightarrow{CA} = \hat{\jmath} - \hat{k}$. No two vectors are II.

The points are non-collinear. $|\overrightarrow{AB}| = |\overrightarrow{BC}| = |\overrightarrow{CA}|$

The points form an equilateral triangle.

2. The lengths of the sides are $\sqrt{2}, \sqrt{2}, \sqrt{2}$.

3. Area $= \frac{\sqrt{3}}{4}a^2 = \frac{\sqrt{3}}{4}(\sqrt{2})^2 = \frac{\sqrt{3}}{2}$.

4. 60°

5. The area of the parallelogram with AB,AC as adjacent sides is

$$= |\overrightarrow{AB} \times \overrightarrow{AC}| = \left\| \begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{matrix} \right\| = |\hat{\imath} + \hat{\jmath} + \hat{k}| = \sqrt{3}\ .$$

III. 1. Area of the parallelogram $= |\overrightarrow{AB} \times \overrightarrow{AD}| = |3\hat{\imath} \times (-4\hat{\jmath})| = 12.$

2. $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = 3\hat{\imath} - 4\hat{\jmath}$; $\overrightarrow{BD} = \overrightarrow{BA} + \overrightarrow{AD} = -3\hat{\imath} - 4\hat{\jmath}$.

Area of the parallelogram with $\overrightarrow{AC}$ and $\overrightarrow{BD}$

as adjacent sides is $= |\overrightarrow{AC}\ \text{X}\ \overrightarrow{BD}| = 24.$

3. a) 1 : 2. [NOTE : this is a theorem in geometry]

4. $(\vec{a} + \vec{b})\ \text{x}\ (\vec{a} - \vec{b}) = (3\hat{\imath} - 4\hat{\jmath})\ \text{X}\ (3\hat{\imath} + 4\hat{\jmath}) = 24\,\hat{k}.$

5. Geometrically speaking, the vectors $(\vec{a} + \vec{b})$ and $(\vec{a} - \vec{b})$ lie on the xoy plane. Therefore, $\pm\,\hat{k}$ are the perpendicular unit vectors. [OR] you can also use the formula for $\pm\,\hat{n}$. Refer, NCERT book.

IV. P,Q and R are the mid-points of AB,BC and CA respectively.

A = (4,2,0), B = (1,3,0), C = (1,1,3).

$P = \left(\frac{5}{2}, \frac{5}{2}, 0\right), Q = \left(1, 2, \frac{3}{2}\right)$ and $R = \left(\frac{5}{2}, \frac{3}{2}, \frac{3}{2}\right)$.

1. Now, $\overrightarrow{CP} = \overrightarrow{OP} - \overrightarrow{OC} = \frac{3}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath} - 3\hat{k}.$

2. Using section formula and the ratio $CG_1 : G_1P = 2:1$, the position Vector of G_1 is $2\hat{\imath} + 2\hat{\jmath} + \hat{k}.$

3. $\overrightarrow{AQ} = \overrightarrow{OQ} - \overrightarrow{OA} = -3\hat{\imath} + \frac{3}{2}\hat{k}.$

4. Using section formula and the ratio $AG_2 : G_2Q = 2:1$, the position

Vector of G_2 is $2\hat{\imath} + 2\hat{\jmath} + \hat{k}$.

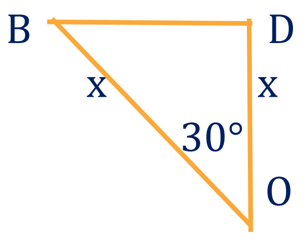
5. Using section formula and the ratio $BG_3 : G_3R = 2:1$, the position

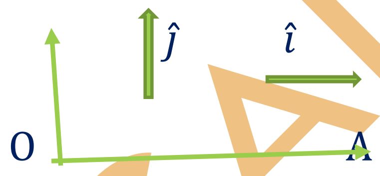
Vector of G_3 is $2\hat{\imath} + 2\hat{\jmath} + \hat{k}$.

V. 1. $\overrightarrow{OA} = x\hat{\imath}$.

[Any vector is length times the unit vector in its direction].

2.





Assuming , a three dimensional view,

consider OD $\perp$ OA, complete tri.ODB.

Now, $\sin 30 = \frac{1}{2} = \frac{BD}{x}$; $\cos 30° = \frac{\sqrt{3}}{2} = \frac{OD}{x}$.

$\overrightarrow{OB} = OD\,\hat{\jmath} + DB\,(-\hat{\imath}) = \quad = -\frac{1}{2}\,x\hat{\imath} + \frac{\sqrt{3}}{2}x\hat{\jmath}$.

3. Similarly, $\overrightarrow{OC} = -\frac{1}{2}\,x\hat{\imath} - \frac{\sqrt{3}}{2}x\hat{\jmath}$

4. $\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = 30\hat{k} - x\hat{\imath}$, $\hat{k}$ is in OP direction.

5. $\overrightarrow{BP} = \overrightarrow{BO} + \overrightarrow{OP} = \frac{1}{2}\,x\hat{\imath} - \frac{\sqrt{3}}{2}x\hat{\jmath} + 30\hat{k}$, using (from) Qn. 2.

END OF CHAPTER 10

// K.MANI , SALEM, TAMILNADU //

XI.THREE DIMENSIONAL GEOMETRY

VERY SHORT ANSWER TYPE

1. Write the vector equation of x-axis.
2. Write the vector equation of y-axis.
3. Write the vector equation of z-axis.
4. Write the cartesian equation of x-axis.
5. Find the cartesian equation of y-axis.
6. Find the cartesian equation of z-axis.
7. Find the point where the perpendicular drawn from (2,-1,4) meets XY plane.
8. Find the point where the perpendicular drawn from (5,-1,6) meets YZ plane.
9. Find the point where the perpendicular drawn from (4,3,7) meets XZ plane.
10. Find the point where the perpendicular drawn from (2,-1,-1) meets the XY plane.
11. Find the distance of the point (-2,8,7) from the xy – plane.
12. Find the distance of the point (-6,5,- 4) from the yz – plane.
13. Find the distance of the point (-2,-6,-7) from the xz – plane.
14. Find the distance of the point (8,0,-9) from the xy – plane.
15. Find the distance of the point (3,-4,0) from the xz – plane.
16. a) Find the distance of the point (3,4,-1) from the x - axis.

    b) Find the distance of the point (0,8,7) from the x – axis.
17. If d_1,d_2,d_3 denote the distances of the point (2,3,7) from the xy,yz,xz planes respectively arrange them in ascending order.

18. A plane passing through the point (2,-1,-3) is drawn parallel to the xy - plane. Find its equation.
19. A plane passing through the point (-2,0,-3) is drawn parallel to the yz - plane. Find its equation.
20. A plane passing through the point (6,-1,-7) is drawn parallel to the xz - plane. Find its equation.
21. The x,y and z intercepts of a plane are 3,-4,4. Find its equation.
22. Find the x – intercept of the plane $2x - y - 7z + 11 = 0$.
23. Find the y – intercept of the plane $3x - 7y + 11 = 0$.
24. Find the z – intercept of the plane $2x + 5y + 3z - 8 = 0$.
25. Find the equation of the plane passing through the origin and parallel to the plane $2x - 7y + 6z + 5 = 0$.
26. Find the equation of the plane passing through the point (3,-2,-5) and perpendicular to x – axis.
27. Find the equation of the plane passing through the point (- 3,7,0) and perpendicular to y – axis.
28. Find the equation of the plane passing through the point (2,-1,4) and perpendicular to z – axis.
29. Find the distance between the planes $x - 2y + 2z = 0$ and $2x - 4y + 4z - 7 = 0$.
30. Write the vector equation of the xy plane in normal form.
31. Write the vector equation of a plane parallel to the xy plane, clearly specifying the assumptions made.
32. Write the vector equation of the yz plane in normal form.

33. Write the vector equation of a plane parallel to the yz plane, clearly specifying the assumptions made.
34. Write the vector equation of the xz plane in normal form.
35. Write the vector equation of a plane parallel to the xz plane, clearly specifying the assumptions made.
36. Write the direction cosines of x – axis.
37. Write the direction cosines of y – axis.
38. Write the direction cosines of z – axis.
39. Do these lines intersect ? If so, at what point ?

    $L_1 : \frac{x}{-3} = \frac{y}{2} = \frac{z}{2}$ and $L_2 : \frac{x}{3} = \frac{y}{1} = \frac{z}{-5}$
40. The direction cosines of the line $\frac{x}{12} = \frac{y}{1} = \frac{z}{-10}$.
41. Find the direction ratios of line passing through the points (2,-3,1) and (3,-4,-5).
42. Find the equation of the line passing through the point (-2,3,4) and perpendicular to yz plane.
43. Find the equation of the line passing through the point (2,0,6) and perpendicular to zx plane.
44. Find the equation of the line passing through the point (-2,-2,1) and perpendicular to xy plane.
45. If α, β *and* γ are the direction angles of a vector then, find the value of $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma$.
46. Write the cartesian equation of the plane containing the lines

    $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$.
47. Find the distance of the plane $\vec{r} \cdot (3\hat{\imath} + 4\hat{\jmath}) = 1$ from the origin.

48. Write the equation of the plane $\vec{r} \cdot 3\hat{\imath} = 1$ in cartesian form.

49. Write the equation of the plane $\vec{r} \cdot (-4\hat{\jmath}) = 5$ in cartesian form.

50. Write the equation of the plane $\vec{r} \cdot \hat{k} = 0$ in cartesian form.

51. Find whether the plane $\vec{r} \cdot (3\hat{\imath} + \hat{\jmath} - \hat{k}) = 2$ contains the line

$\vec{r} = (2\hat{\imath} - 3\hat{\jmath} - \hat{k}) + \lambda(\hat{\imath} - \hat{\jmath} + 2\hat{k})$.

52. Find the point where the plane $3x + 4y + 5z = 8$ cuts the x – axis.

53. Find the distance of the point (-1,-5,-10) from the point where the plane $2x - 8y + 4z - 5 = 0$ cuts the x – axis.

54. Find the distance of the point (-1,-5,-10) from the point where the plane $2x + 4z - 5 = 0$ cuts the z – axis.

55. Find the distance of the point (-1,-5,-10) from the point where the plane $8y - 5 = 0$ cuts the y – axis.

56. The equation of a line is L : $\vec{r} = (2\hat{\imath} + 3\hat{\jmath} - 5\hat{k}) + \lambda(-3\hat{\jmath} + 4\hat{k})$. If the point (p, q, -1) lies on L , find p and q.

57. Find the shortest distance of the origin from the intersection of the planes $x = 3, y = 4$.

58. If the line $\frac{x+1}{-2} = \frac{y+2}{3} = \frac{z+5}{4}$ lies on the plane $x + my - z = 0$, find the value of m.

59. Where does line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ intersect the plane $2x + 3y - z = 9$?.

60. A line passes through the points (2,3,4) and (0,0,4). Find its distance from the origin.

61. If plane $x + 2y - z = 0$ contains the line $\frac{x+1}{-2} = \frac{y+2}{m} = \frac{z+5}{4}$, find m.

62. Write the direction cosines of a line parallel to the x – axis.

63. Write the direction cosines of a line parallel to the y – axis.

64. Write the direction cosines of a line parallel to the z – axis.

65. Write the direction cosines of a line equally inclined to the axes.

66. Find the direction cosines of a diagonal of the square of side "a" units drawn in the xy-plane with the co-ordinate axes as its adjacent sides.

67. If a vector $\overrightarrow{OL}$ makes angles $90°, 135°$ and $45°$ respectively with the co-ordinate axes, find the angles made by $-\overrightarrow{OL}$ with co-ordinate axes.

68. The equations of a line passing through the points A and B are $\frac{x-1}{-2} = \frac{y-2}{1} = \frac{z-2}{2}$. If the point A = (1,2,2) , find the co-ordinates of the point B.

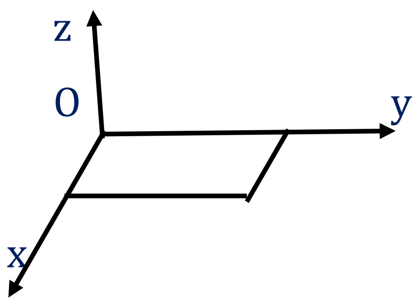
69. Find the equation of the plane passing through the point (1,3,1) and parallel to the plane 2x + 4y – 5z = 2.

70. A vector makes angle π with the positive direction of x – axis. What are the angles it makes with the y,z axes ?

XI. THREE DIMENSIONAL GEOMETRY

[CASE STUDY BASED M C QS]

I. A plane π_1 is in the form of a rectangle. Its sides pass through the points (0,0,0), (5,0,0) and (0,6,0). Now, answer the following questions.



1. If one of its diagonal vector is $8\hat{\imath} + 11\hat{\jmath}$, then the length of the diagonal is

a) $\sqrt{185}$ b) $\sqrt{61}$ c) $\frac{8}{\sqrt{185}}$ d) $\frac{11}{\sqrt{185}}$

2. A normal vector to the plane π_1 is

a) $\hat{\imath}$ b) $\hat{k}$ c) $\hat{\imath} - \hat{\jmath}$ d) $\hat{\imath} + \hat{\jmath}$.

3. The equation of the plane π_2 drawn passing through this diagonal and perpendicular to the plane π_1 is

a) x – y = 0   b) 8x – 11y = 0   c) 11x – 8y = 0   d) 11x + 8y = 0.

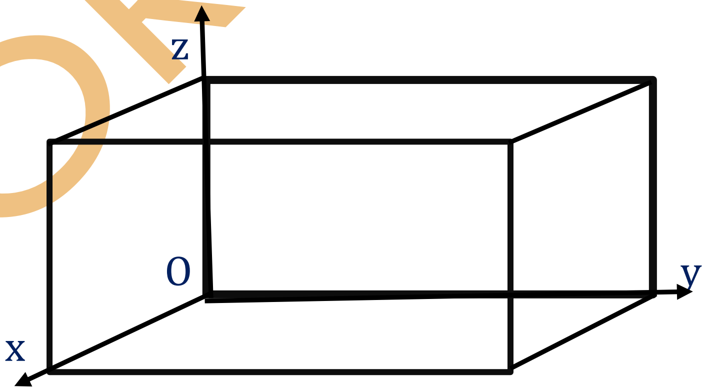
4. The area of the planar region π_1 is

a) 30   b) 44   c) 66   d) 88.

5. The unit vectors normal to the plane π_2 are

a) $\pm\frac{11\hat{\imath}+8\hat{\jmath}}{\sqrt{185}}$ b) $\pm\frac{11\hat{\imath}-8\hat{\jmath}}{\sqrt{185}}$ c) $\pm\frac{8\hat{\imath}-11\hat{\jmath}}{\sqrt{185}}$ d) $\pm\hat{k}$

II. A rectangular box in the first Octant is bounded by the planes x = 1, y = 3, z = 2. Now, answer the following questions.



1. The vector equation of the plane (in normal form) parallel to x = 1 is

a) $\hat{r}\cdot\hat{\imath} = 0$ b) $\hat{r}\cdot\hat{\jmath} = 0$ c) $\hat{r}\cdot\hat{\imath} = 1$ d) $\hat{r}\cdot\hat{k} = 0$.

2. The direction cosines of the normal to the plane parallel to y = 3 are

a) (0,1,0)   b) (0,0,1)   c) (1,0,0)   d) (0,1,-1) .

3. The normal vector to the plane parallel to z = 2 is

a) $\hat{\imath}$ b) $-\hat{\jmath}$ c) - $\hat{k}$ d) $\hat{r} + \hat{\jmath}$.

4. The vector equation of the diagonal of the box passing through O is

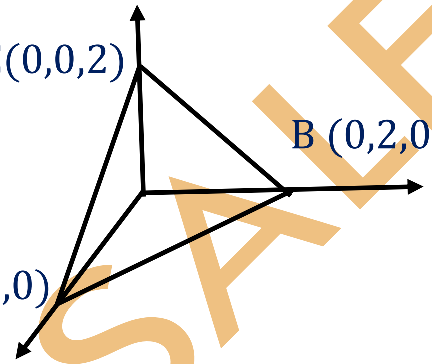
a) $\vec{r} = \lambda(\hat{\imath} + 3\hat{\jmath} + 2\hat{k})$ b) $\vec{r} = \hat{\imath} + 3\hat{\jmath} + 2\hat{k}$

c) $\vec{r} = \lambda(\hat{\imath} - 3\hat{\jmath} - 2\hat{k})$ d) $\vec{r} = \lambda(\hat{\imath} + 3\hat{\jmath} - 2\hat{k})$

5. What is the radius of the largest sphere that is packed in the box ?

a) 1 b) 3 c) 2 d) $\sqrt{14}$

III. In the adjoining diagram, a plane In the form of a triangle is shown Intercepting the co-ordinate axes at A, B and C respestively.



Now, answer the following questions :

1. The equation of the plane is

a) $x + y + z = 2$ b) $2x + 2y + 2z = 1$

c) $x + y + z = 1$ d) $x + y + z + 2 = 0$.

2. The distance of the plane from the origin is

a) $\sqrt{3}$ b) $\frac{2}{3}$ c) $\sqrt{\frac{2}{3}}$ d) $\frac{2}{\sqrt{3}}$

3. A normal vector to the plane is

a) $\hat{\imath} + \hat{\jmath} + \hat{k}$ b) $\hat{\imath} - \hat{\jmath} + \hat{k}$ c) $\hat{\imath} - \hat{\jmath} - \hat{k}$ d) $\hat{\imath} - 2\hat{\jmath} + \hat{k}$.

4. The area of the triangle ABC is

a) $|(2\hat{\imath} + 2\hat{\jmath}) \times (-2\hat{\imath} - 2\hat{k})|$ b) $\frac{1}{2}|(2\hat{\imath} - 2\hat{\jmath}) \times (2\hat{\imath} + 2\hat{k})|$

c) $\frac{1}{2}|(-2\hat{\imath} + 2\hat{\jmath}) \times (-2\hat{\imath} + 2\hat{k})|$ d) $|(-2\hat{\imath} + 2\hat{\jmath}) \times (2\hat{\imath} + 2\hat{k})|$

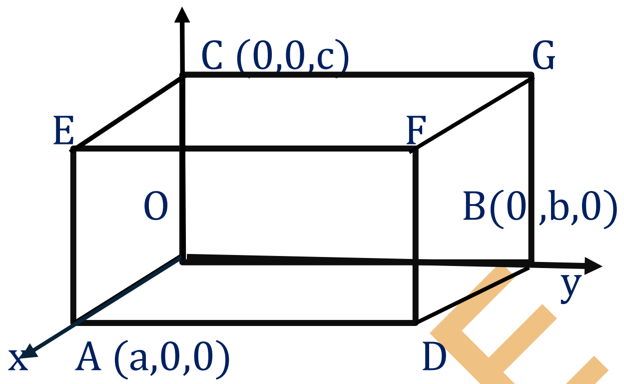
5. The equation of the plane passing through the origin and perpendicular to this plane is

a) $x - 2y + z = 0$ b) $x - y + z = 0$

c) $x + y - z = 0$ d) $x + 2y + z = 0$.

IV. Study the adjoining diagram and answer the following questions.

In the figure, ADFEOBGC is a cuboid whose three co-terminous edges lie along the co-ordinate axes.



1. The direction cosines of the edge GF are

   a) (-1,0,0) b) (0,1,0) c) (0,0,1) d) (1,0,0).

2. The direction ratios of the diagonal OF are

   a) (a,b,-c) b) (1,1,1) c) (a,b,c) d) (-a,b,c).

3. The vector equation of the edge BG is

   a) $\vec{r} = b\hat{j} + \lambda\, c\hat{k}$ b) $\vec{r} = (a\hat{\imath} + b\hat{j} + c\hat{k}) + \lambda\, c\hat{k}$

   c) $\vec{r} = \lambda\, c\hat{k}$ d) $\vec{r} = b\hat{j} + c\hat{k}$

4. The cartesian equation of the face ADFE is

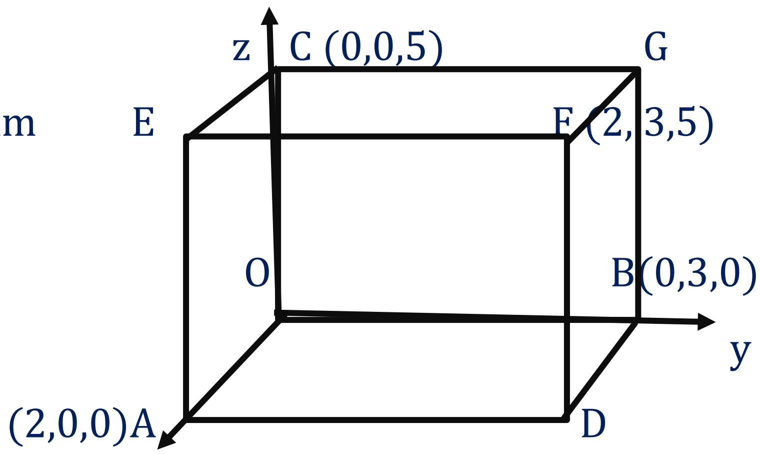
   a) y = 0 b) x = a c) z = a d) y = a.

5. The cartesian equation of the diagonal OG is

   a) $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ b) $\frac{x}{0} = \frac{y}{-b} = \frac{z}{-c}$

   c) $\frac{x}{0} = \frac{y}{b} = \frac{z}{c}$ d) $\frac{x}{0} = \frac{y-b}{b} = \frac{z-c}{c}$.

V. Study the adjoining diagram and answer the following Questions.



1. Which of the following pair of lines passing through the indicated points are skew lines ?

   a) OA,FG b) BC,DE c) AF,OB d) CF,OD.

2. The distance between the lines AB and EG is

   a) 5 b) 3 c) 2 d) $\sqrt{38}$.

3. The centre of the cuboid from the origin is

   a) $\sqrt{38}$ b) $2\sqrt{38}$ c) $\sqrt{19}$ d) $\sqrt{38}/2$

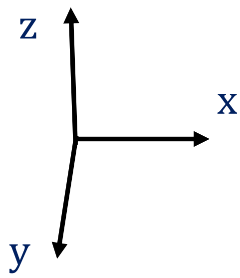
4. The distance between the skew lines OE and DG is

   a) 3 b) 2 c) 5 d) $\sqrt{38}$.

5. The projection of AF on z-axis is

   a) $\frac{5}{\sqrt{34}}$ b) 5 c) 3 d) $\frac{3}{\sqrt{34}}$.

VI. A balloon flying over flatland reports its position at 7.40 a.m. as (7.8 , 5.4 , 1.2), the coordinates being given in kilometers relative to a checkpoint on the ground. By 7.50 a.m. its position has changed to (9.3 , 4.4 , 0.7). Assume that it continues to descend at the same speed along the same line.



The diagram, only a fairly suggestive, is given to aid your understanding.

x-axis – horizontal direction,

y-axis – towards the observer(here, the student),

z-axis – vertical direction.

Now, answer the following questions.

1. The time required to land after 7.50 a.m. is

   a) 14 min   b) 4 min   c) 12 min   d) 9 min

2. The cartesian equations of the line along which the balloon moves is

   a) $\frac{x-7.8}{1.5} = \frac{y-5.4}{-1} = \frac{z-1.2}{0.5}$ b) $\frac{x-7.8}{1.5} = \frac{y-5.4}{-1} = \frac{z-1.2}{-0.5}$

   c) $\frac{x-7.8}{3} = \frac{y-5.4}{2} = \frac{z-1.2}{-1}$ d) $\frac{x-7.8}{3} = \frac{y-5.4}{-2} = \frac{z-1.2}{1}$

3. The distance covered along the z axis direction is

   a) 0.7 km   b) 1.2 km   c) 1.93 km   d) 0.5 km

4. The distance covered along the x axis direction is

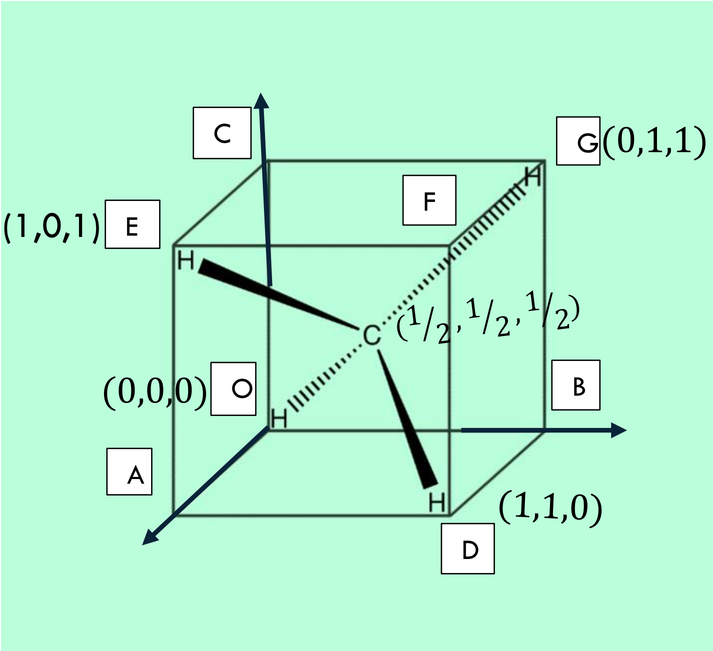
   a) 3.4 km   b) 1.4 km   c) 1.5 km   d) 3.6 km

5. The distance covered along the y axis direction is

   a) 3.4 km   b) 1.4 km   c) 1 km   d) 2 km.

VII. In a methane molecule, a carbon atom is surrounded by four hydrogen atoms. Assume that the hydrogen atoms are at (0,0,0), (1,1,0), (1,0,1) and (0,1,1) and the carbon atom is at (1/2,1/2,1/2).

Let C denote the carbon atom and H_0, the hydrogen atom at the origin and H_1, H_2, H_3 denote the other hydrogen atoms.



1. The angle measured from a hydrogen atom to carbon atom to hydrogen atom is called the **Bond angle.** The bond angle is

   a) $\sin^{-1}\left(\frac{1}{3}\right)$ b) $\cos^{-1}\left(-\frac{1}{3}\right)$ c) $\cos^{-1}\left(\frac{1}{3}\right)$ d) $\sin^{-1}\left(\frac{-1}{3}\right)$

2. The distance between two hydrogen atoms is

   a) 2 b) 4 c) $\sqrt{2}$ d) $2\sqrt{2}$.

3. The distance between the carbon atom and a hydrogen atom is

   a) ¾ b) $\sqrt{3}/4$ c) $\sqrt{3}/2$ d) 3/2.

4. Two skew lines containing hydrogen atom(s) are

   a) OA,FG b) BC,DE c) AF,OB d) DG,OE.

5. The distance between the skew lines is

   a) 1 b) 2 c) $\sqrt{2}$ d) 3/2.

CHAPTER XI - SOLUTIONS

1. $\vec{r} = \vec{0} + \lambda\,\hat{\imath}$.

   [The x-axis passes through the origin (0,0,0) and $\hat{\imath}$ is the unit vector along the positive direction of x-axis].

2. $\vec{r} = \vec{0} + \lambda\,\hat{j}$.

[The y-axis passes through the origin (0,0,0) and $\hat{j}$ is the unit vector along the positive direction of y-axis].

3. $\vec{r} = \vec{0} + \lambda\,\hat{k}$.

[The z-axis passes through the origin (0,0,0) and $\hat{k}$ is the unit vector along the positive direction of z-axis].

4. The x-axis, passes through (0,0,0) and Direction cosines (dcs) are (1,0,0).

$\therefore \frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0}$ or $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$.

5. The y-axis, passes through (0,0,0) and Direction cosines (dcs) are (0,1,0).

$\therefore \frac{x-0}{0} = \frac{y-0}{1} = \frac{z-0}{0}$ or $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$.

6. The z-axis, passes through (0,0,0) and Direction cosines (dcs) are (0,0,1).

$\therefore \frac{x-0}{0} = \frac{y-0}{0} = \frac{z-0}{1}$ or $\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$.

NOTE : IN THREE DIMENSIONAL SPACE, TO REACH A POINT (a,b,c), STARTING FROM THE ORIGIN WE MOVE x UNITS ALONG x AXIS, y UNITS ALONG y AXIS AND z UNITS ALONG z AXIS.(THE POSITIVE/NEGATIVE DIRECTION OF AXIS IS CHOSEN APPROPRIATELY)

7. Perpendicular from (2,-1,4) means no movement in the z – direction.

∴ The point in the xy plane is (2,-1,0).

8. Perpendicular from (5,-1,6) means no movement in the x – direction.

∴ The point in the yz plane is (0,-1,6).

9. Perpendicular from (4,3,7) means no movement in the y – direction.

∴ The point in the zx plane is (4,0,7).

10. Perpendicular from (0,-1,-1) means no movement in the z – direction.

∴ The point in the xy plane is (0,-1,0).

11. Distance of a point from xy-plane is | z co-ordinate| = 7.

12. Distance of a point from yz-plane is | x co-ordinate| = |-6|= 6.

13. Distance of a point from xz-plane is | y co-ordinate| = |-6|= 6.

14. Distance = 9.

15. Distance = 4.

16. a) To find the distance of the point A (3,4,-1) from the x – axis, draw a perpendicular to x-axis from A, meeting it at (3,0,0) = B, say.

$\therefore \ Required\ distance\ AB = \sqrt{0^2 + 4^2 + (-1)^2} = \sqrt{17}$.

b) The distance of the point (0,8,7) from the x – axis means from (0,0,0).

$\therefore \ Required\ distance = \sqrt{0^2 + 8^2 + (7)^2} = \sqrt{113}$.

17. d_1 = distance of the point from xy plane

= |z co-ordinate| = 7. Similarly, d_2 = 2, d_3 = 3.

$\therefore$ $2 < 3 < 7 \Rightarrow \ d_2 < d_3 < d_1$.

NOTE : IF A PLANE PASSES THROUGH A POINT AND IS PARALLEL TO xy- PLANE, ITS EQUATION IS z = z CO-ORDINATE OF THE POINT.

18. Parallel to xy – plane means, the normal vector is $\hat{k}$.

the dcs are 0,0,1 → A,B,C.

The plane passes through the point (2,-1,-3) =(x_1, y_1, z_1)

The equation is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$. ∴ z + 3 = 0.

OR, since the plane passes thro' (2,-1,-3) and parallel to xy – plane,

Its equation is z = z co-ordinate. i.e. z = -3.

19. x = -2 . Refer solution to problem 18.

20. y = -1. Refer solution to problem 18.

21. The intercept form of equation of a plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. Sub,

a = 3, b = - 4, c = 4 and simplifying, the eqn is 4x – 3y + 3z = 12.

22. 2x – y – 7z + 11 = 0. Rewriting, 2x – y – 7z = -11 , dividing by – 11 and Writing in intercept form, we get $\frac{x}{-11/2}+\frac{y}{11}+\frac{z}{11/7}=1.$

Therefore, x – intercept is - 11/2.

OR, x-intercept is the where y = z = 0. Letting y = z = 0 in the equation of the plane, we get 2x + 11 = 0 ⇒ x = -11/2.

23. The y- intercept is 11/7. [Putting x = 0, z = 0(of course no z term)].

24. The z – intercept is 8/3.

25. The equation of the plane parallel to 2x – 7y + 6z + 5 = 0 is 2x – 7y + 6z + k = 0. This passes thro' (0,0,0). So, sub x = y = z = 0, We get k = 0. Therefore, the equation is 2x – 7y + 6z = 0.

26. The plane passes thro' (3,-2,-5)=(x_1, y_1, z_1). Perpendicular to xaxis.

∴ The normal vector to the plane is $\hat{\imath}$. The dcs are 1,0,0 → A,B,C.
Using the equation $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0,$

Sub and simplifying, the equation is x – 3 = 0.

OR, since the plane passes thro' (3,-2,-5) and perpendicular to x - axis,

Its equation is x = x co-ordinate. i.e. x = 3.

27. The equation is y = 7.

28. The equation is z = 4.

29. Equation 1 → 2x – 4y + 4z = 0. Equation 2 is 2x – 4y + 4z – 7 = 0.

Distance between two parallel planes ax+by+cz + d_1 = 0 and ax+by+cz + d_2 = 0 is $\left|\frac{d_1 - d_2}{\sqrt{a^2+b^2+c^2}}\right|$. Here, a=2,b=-4,c=4,d_1=0,d_2=-7.

Sub and simplifying , the distance is 7/6.

30. The normal form of plane is $\vec{r}\cdot\hat{n} = p\ (or)\ \vec{r}\cdot\vec{n} = q$.

The (unit) normal vector to xy plane is $\hat{k}$. The xy-plane passes thro' the Origin and hence, p = distance of the plane from the origin = 0.

∴ The equation of xy-plane is $\vec{r} \cdot \hat{k} = 0$.

[$-\hat{k}$ is also normal to xy-plane and hence can be used]

31. Any plane parallel to the xy-plane has two known (unit) normal vectors, $\hat{k}$ *and* $-\hat{k}$. If the plane is at a distance p from the origin, then its equation is $\vec{r} \cdot \hat{k} = p$ *or* $\vec{r} \cdot (-\hat{k}) = p$.

32. The equation of yz-plane is $\vec{r} \cdot \hat{\imath} = 0$.

33. Any plane parallel to the yz-plane has two known (unit) normal vectors, $\hat{\imath}$ *and* $-\hat{\imath}$. If the plane is at a distance p from the origin, then its equation is $\vec{r} \cdot \hat{\imath} = p$ *or* $\vec{r} \cdot (-\hat{\imath}) = p$.

34. The equation of xz-plane is $\vec{r} \cdot \hat{\jmath} = 0$.

35. Any plane parallel to the xz-plane has two known (unit) normal vectors, $\hat{\jmath}$ *and* $-\hat{\jmath}$. If the plane is at a distance p from the origin, then its equation is $\vec{r} \cdot \hat{\jmath} = p$ *or* $\vec{r} \cdot (-\hat{\jmath}) = p$.

36. The direction angles of x - axis are 0°, 90°, 90°. Therefore, the dcs are $\cos 0°, \cos 90°, \cos 90°$. That is 1,0,0.

37. The direction angles of y - axis are 90°, 0°, 90°. Therefore, the dcs are $\cos 90°, \cos 0°, \cos 90°$. That is 0,1,0.

38. The direction angles of z - axis are 90°, 90°, 0°. Therefore, the dcs are $\cos 90°, \cos 90°, \cos 0°$. That is 0,0,1.

39. ***DO IT YOURSELF.*** [ANS : The two lines intersect at the origin(0,0,0)].

40. The parallel vector to the line is $12\,\hat{\imath} + \hat{\jmath} - 10\,\hat{k}$. The drs are 12,1,-10. Taking $\sqrt{12^2 + 1^2 + (-5)^2} = \sqrt{245}$ and dividing, the dcs are $\pm\frac{12}{\sqrt{245}}, \pm\frac{1}{\sqrt{245}}, \pm\frac{-10}{\sqrt{245}}$.

41. The points (2,-3,1) and (3,-4,-5). The drs are 3-2,-4+3,-5-1. i.e., 1,-1,-6.

42. The line passes thro' (-2,3,4) and is perpendicular to yz-plane.

$\therefore$ The parallel vector to the line is $\hat{\imath}$. Its dcs are (1,0,0).

The equation is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$. i.e., $\frac{x+2}{1} = \frac{y-3}{0} = \frac{z-4}{0}$.

43. The line passes thro' (2,0,6) and is perpendicular to xz-plane.

$\therefore$ The parallel vector to the line is $\hat{\jmath}$. Its dcs are (0,1,0).

The equation is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$. i.e., $\frac{x-2}{0} = \frac{y-0}{1} = \frac{z-6}{0}$.

44. The line passes thro' (-2,-2,1) and is perpendicular to xy-plane.

$\therefore$ The parallel vector to the line is $\hat{k}$. Its dcs are (0,0,1).

The equation is $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$. i.e., $\frac{x+2}{0} = \frac{y+2}{0} = \frac{z-1}{1}$.

45. If α, β *and* γ are the direction angles of a vector then, direction cosines are $cos\alpha, cos\beta, cos\gamma$. Also we know that , $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

$\Rightarrow$ 1 - $\sin^2\alpha$ + 1 - $\sin^2\beta$ + 1 - $\sin^2\gamma = 1$

$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2.$

46. The first line passes thro' the point A (x_1, y_1, z_1) and has parallel vector $a_1\hat{\imath} + b_1\hat{\jmath} + c_1\hat{k} = \vec{u}$.The second line passes thro' the point B(x_2, y_2, z_2) and has parallel vector $a_2\hat{\imath} + b_2\hat{\jmath} + c_2\hat{k} = \vec{v}$.

Two points on the plane and two parallel vectors to the plane are known.

1) Use the two points A and B and one of the parallel vectors $\vec{u}$ *or* $\vec{v}$; or

2) Use one point A or B and the two parallel vectors $\vec{u}$ *or* $\vec{v}$; For example,

The cartesian equation is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$. etc.,

47. The equation of plane is $\vec{r}\cdot(3\hat{\imath} + 4\hat{\jmath}) = 1$. This is in the form $\vec{r}\cdot\vec{n} = q$.

The normal vector to the plane is $(3\hat{\imath} + 4\hat{\jmath})$ whose modulus is 5.

Dividing the equation by 5, $\vec{r}\cdot\frac{(3\hat{\imath}+4\hat{\jmath})}{5} = \frac{1}{5}$, which is in the form $\vec{r}\cdot\hat{n} = p$,

Where p = 1/5 is the distance of the plane from the origin.

48. Given, $\vec{r} \cdot 3\hat{\imath} = 1$. Sub, $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$, we get $(x\vec{\imath} + y\vec{\jmath} + z\vec{k}) \cdot 3\hat{\imath} = 1$,

    The equation is 3x = 1.

49. Given, $\vec{r} \cdot (-4\hat{\jmath}) = 5$. Proceeding as above, the eqn is - 4y =5 or 4y+5=0.

50. $\vec{r} \cdot \hat{k} = 0$. The cartesian equation is z = 0.

51. To find whether a plane contains a line,(i) check that the point on the line lies on the plane also, (ii) and that the normal vector to the plane and the parallel vector to the line are perpendicular.

    The line passes through the point $2\hat{\imath} - 3\hat{\jmath} + \hat{k} = \vec{a}$. Sub, $\vec{a}$ in the place of $\vec{r}$ in the equation of the plane $(2\hat{\imath} - 3\hat{\jmath} + \hat{k}) \cdot (3\hat{\imath} + \hat{\jmath} - \hat{k}) = 2$ is true.

    The point lies on the plane.

    Now, normal vector to the plane is $\vec{n} = (3\hat{\imath} + \hat{\jmath} - \hat{k})$ and the parallel vector to the line is $\vec{b} = (\hat{\imath} - \hat{\jmath} + 2\hat{k})$. Now, $\vec{n} \cdot \vec{b} = 0$.

    Therefore, the line lies on the plane.

52. At the point, where the plane cuts the x-axis, y = z = 0.Sub, x = 8/3.

    Therefore, the point is (8/3,0,0).

53. The given point is(-1,-5,-10).The point where the plane cuts x-axis is (5/2,0,0). Distance $= \frac{1}{2}\sqrt{549}$.

54. Given point (-1,-5,-10). The point where the plane cuts z-axis is (0,0,5/4) Use distance formula.

55. The given point is(-1,-5,-10). The point where the plane cuts y-axis is (0,5/8,0). Use distance formula.

56. The PV of the (p,q,-1) is $p\hat{\imath} + q\hat{\jmath} - \hat{k}$. This lies on the line L. So, sub for $\vec{r}$,

    And comparing the corresponding components , we get

    p = 2 , q = 3 - 3λ, -1 = -5 + 4λ $\Rightarrow \lambda = 1$ *and* $q = 0$.

57. ***PLEASE TRY YOURSELF.*** [ANS : 5]

58. Since the line lies on the plane, the normal vector to the plane and the parallel vector to the line are perpendicular.

The normal vector to the plane is $\vec{n} = \hat{\imath} + m\hat{\jmath} - \hat{k}$ and the parallel vector to the line is $\vec{b} = -2\hat{\imath} + 3\hat{\jmath} + 4\hat{k}$. So, $\vec{n} \cdot \vec{b} = 0 \Rightarrow$ m = 2.

59. Letting $\frac{x}{2} = \frac{y}{3} = \frac{z}{4} = \lambda \Rightarrow x = 2\lambda, y = 3\lambda, z = 4\lambda$.

Sub, in the plane's equation $\lambda = 1$. The point is (2,3,4).

60. ***PLEASE TRY YOURSELF.*** [ANS : 4].

61. ***PLEASE TRY YOURSELF.*** [ANS : 3]

62. The dcs of a line parallel to x-axis are same as those of the x-axis itself. So, the dcs are 1,0,0.

63. The dcs of a line parallel to y-axis are same as those of the y-axis itself. So, the dcs are 0,1,0.

64. The dcs of a line parallel to z-axis are same as those of the z-axis itself. So, the dcs are 0,0,1.

65. Since a line is equally inclined to the co-ordinate axes, let α, α, α be the direction angles. Then, we have

$$\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1.$$ Proceed.

66. Let OABC be the square whose two adjacent sides are along positive x,y axes. $\overrightarrow{OB} = a\hat{\imath} + a\hat{\jmath}$, $\overrightarrow{AC} = -a\hat{\imath} + a\hat{\jmath}$, are the diagonals.

$|\overrightarrow{OB}| = |\overrightarrow{AC}| = \sqrt{2}\,a$. So, dcs are $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ or $\left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

67. The dcs are $\cos 90°, \cos 135°, \cos 45°$. i.e., $\left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

So, the unit vector along $\overrightarrow{OL}$ *is* $\frac{-1}{\sqrt{2}}\hat{\jmath} + \frac{1}{\sqrt{2}}\hat{k}$.

So, the unit vector along $-\overrightarrow{OL}$ *is* $\frac{1}{\sqrt{2}}\hat{\jmath} - \frac{1}{\sqrt{2}}\hat{k}$.

So, the angles are 90°,45°,135°.

68. $\frac{x-1}{-2} = \frac{y-2}{1} = \frac{z-2}{2}$ is the equation of a line passing thro' A & B.

$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$ is the equation of a line intwo point form.

A = (1,2,2) = (x_1, y_1, z_1) . Let B = (x_2, y_2, z_2). Sub $x_1 = 1, y_1 = 2, z_1 = 2$ and comparing the equations we get B = (x_2, y_2, z_2) = (-1,3,4).

69. The given plane is , say π_1 : 2x + 4y – 5z - 2 = 0.

Any plane parallel to π_1 is 2x + 4y – 5z + k = 0. This passes thro'(1,3,1).

Sub,k = -9. So, the equation of the parallel plane is 2x + 4y – 5z – 9 = 0.

70. A vector makes an angle π with positive direction of x-axis.

So, the vector is along negative direction of x-axis or parallel to that direction. So, it makes 90°, 90° with the y, z axes.

XI - SOLUTIONS TO CASE STUDIES

I. 1. Length of the diagonal = $\sqrt{8^2 + 11^2} = \sqrt{185}$.

2. $\hat{k}$ *or* $-\hat{k}$.

3. The plane passes thro' the points(0,0,0),(8,11,0) and say (8,11,1).

Then, the equation is $\begin{vmatrix} x & y & z \\ 8 & 11 & 0 \\ 8 & 11 & 1 \end{vmatrix} = 0 \Rightarrow 11x - 8y = 0.$

4. d) 88

5. b) The normal vector to the plane π_2 is $11\hat{\imath} - 8\hat{\jmath}$. Now, proceed.

II. 1. The CE of plane parallel to x = 1 is x = 0. The VE is $\vec{r} \cdot \hat{\imath} = 0$.

2. The plane parallel to y = 3 is y = 0.

The normal vector is $\hat{\jmath}$ or $0\hat{\imath} + \hat{\jmath} + 0\hat{k}$.

The drs are 0,1,0. The dcs are 0,1,0.

3. The plane parallel to z = 2 is z = 0. The normal vector is $\hat{k}$ *or* $-\hat{k}$.

4. The diagonal of the box passes thro' (0,0,0) and (1,3,2).

The parallel vector is $\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$. The vector equation is

$\vec{r} = (0\hat{\imath} + 0\hat{\jmath} + 0\hat{k}) + \lambda(\hat{\imath} + 3\hat{\jmath} + 2\hat{k})$. i.e., $\overrightarrow{r} = \lambda(\hat{\imath} + 3\hat{\jmath} + 2\hat{k})$.

5. The radius of the largest sphere is r = 1.

III. 1. $\frac{x}{2} + \frac{y}{2} + \frac{z}{2} = 1 \Rightarrow x + y + z = 2.$

2. The plane is $x + y + z - 2 = 0. \therefore\ Distance = \left|\frac{0+0+0-2}{\sqrt{1^2+1^2+1^2}}\right| = \frac{2}{\sqrt{3}}.$

3. A normal vector to the plane is $\hat{\imath} + \hat{\jmath} + \hat{k}$.

4. c)

5. A normal vector to the plane is $\hat{\imath} + \hat{\jmath} + \hat{k}$.

A perpendicular vector to this is $\hat{\imath} - 2\hat{\jmath} + \hat{k}$. passes thro' (0,0,0).

The equation is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$

That is, 1.(x – 0) + (-2)(y – 0) + 1.(z – 0) = 0 . i.e., x – 2y + z = 0.

IV. 1. AF,OB.

2. AB is parallel to EG. So, distance is 5.

3. The centre of the cuboid is (1,3/2,5/2). Distance = $\sqrt{185}$ /2.

4. The distance between OE and DG is the same as the distance between the planes. So, distance is 3.

5. $\overrightarrow{AF} = 3\hat{\jmath} + 5\hat{k}$. z-axis is $\hat{k}$. $Projection = \frac{(3\hat{\jmath}+5\hat{k})\cdot(\hat{k})}{|\hat{k}|} = 5$.

V. 1. The dcs are 1,0,0.

2. The point F is (a,b,c). $\overrightarrow{OF} = (a\hat{\imath} + b\hat{\jmath} + c\hat{k})$. $|\overrightarrow{OF}| = \sqrt{a^2 + b^2 + c^2}$ = r.

The dcs are a/r,b/r,c/r.

3. The edge BG passes thro' (0,b,0) and (0,b,c). The parallel vector is $0\hat{\imath} + 0\hat{\jmath} + c\hat{k}$. So, the VE is $\vec{r} = b\hat{\jmath} + \lambda . c\hat{k}$

4. The face ADFE : x = a.

5. The diagonal OG passes thro' O(0,0,0) and G(0,b,c).

The CE is $\frac{x-0}{0-0} = \frac{y-0}{b-0} = \frac{z-0}{c-0}$. (or) $\frac{x}{0} = \frac{y}{b} = \frac{z}{c}$.

VI. GIVEN : THE CO-ORDINATES OF O = (7.8 , 5.4 , 1.2)

THE CO-ORDINATES OF A = (9.3 , 4.4 , 0.7)

In 10 min., x increases by 1.5km (1500 m) ,

y decreases by 1km (1000m),

z decreases by 0.5 km (500 m).

In 10 min. z decreases by 0.5 km = 500 m.

In 1 min. z decreases by 50 m.

To reach the ground z must be 0.

So, the time required to cover (z = 700 m = 50 x 14) is 14 min.

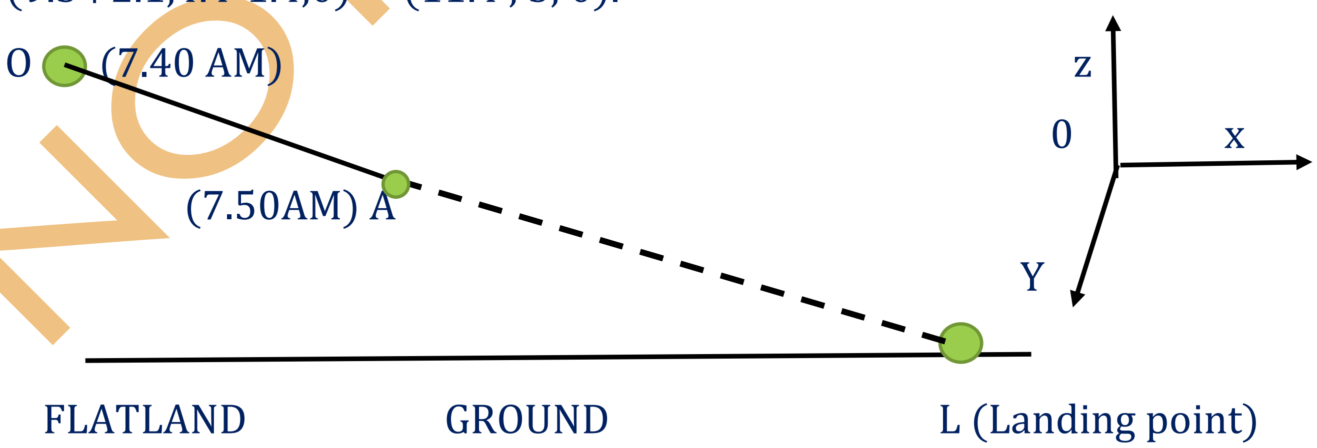
Hence, at 7.50 + 14 min = 8.04 , the balloon will reach the ground.

In 1 min. y decreases by 100 m. So, in 14 min., y will decrease 1400 m.

In 1 min. x increases by 150 m. So, in 14 min., x will increase 2100 m.

Therefore, the coordinates of the landing point are

(9.3+2.1,4.4-1.4,0) = (11.4 , 3, 0).



1.The time required to land after 7.50 AM is 14 minutes.

2. b)

3. Distance covered along z direction(after 7.50 AM) = 0.7 km.

4. Distance covered along x direction(after 7.50 AM) = 2.1 km.

5. Distance covered along y direction(after 7.50 AM) = 1.4 km.

VII. $C = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. Let us denote the hydrogen atoms by

$H_0 = (0,0,0)$, $H_1 = (1,1,0)$, $H_2 = (1,0,1)$ and $H_3 = (0,1,1)$.

1. $\overrightarrow{OC} = \frac{1}{2}\vec{\imath} + \frac{1}{2}\vec{\jmath} + \frac{1}{2}\vec{k}$, $\overrightarrow{OH_1} = \hat{\imath} + \hat{\jmath}$. $\therefore$ $\overrightarrow{CH_1} = \frac{1}{2}\vec{\imath} + \frac{1}{2}\vec{\jmath} - \frac{1}{2}\vec{k}$.

   Similarly, $\overrightarrow{CH_3} = -\frac{1}{2}\vec{\imath} + \frac{1}{2}\vec{\jmath} + \frac{1}{2}\vec{k}$. Let α be the H – C – H angle.

   Then, $\cos\alpha = \dfrac{\frac{1}{2}\cdot\frac{-1}{2} + \frac{1}{2}\cdot\frac{1}{2} + \frac{-1}{2}\cdot\frac{1}{2}}{\sqrt{\frac{3}{4}}\sqrt{\frac{3}{4}}} = \frac{-1}{3} \Rightarrow \alpha = \cos^{-1}\left(\frac{-1}{3}\right)$

2. As any two hydrogen H – H atoms lie at the ends of a diagonal of the face of the cube, Distance = $\sqrt{2}$.

3. c) $\sqrt{3}/2$

4. d) DG,OE

5. a) 1.

END OF CHAPTER 11

// K.MANI , SALEM, TAMILNADU //

XII. LINEAR PROGRAMMING

VERY SHORT ANSWER TYPE

1. Define a convex set.
2. The corner points of a feasible region are the points of intersection of The graphs of constraints with the co-ordinate axes. True or false ? If false, state the reason.
3. Can $Z = 2x^2 + 3y^2$ be the objective of a LPP ?
4. Can $Z = \frac{2}{x} + \frac{3}{y}$ be the objective function of a LPP ?
5. Every LPP has a unique solution. True or false. Support your reason.
6. Every LPP has a solution. True or false. Support your reason.
7. Can $U = 3x + 2y + 5z$ be an objective function of a LPP ?
8. Can an objective function have an optimal solution subject to $|x + y| < 5$?
9. Can an objective function have an optimal solution subject to $|x + y| < 5$, where $x \geq 0, y \geq 0$.
10. Can the following be considered a LPP ? Reason.

    Maximise $Z = 4x + 11y$, subject to constraint 1,

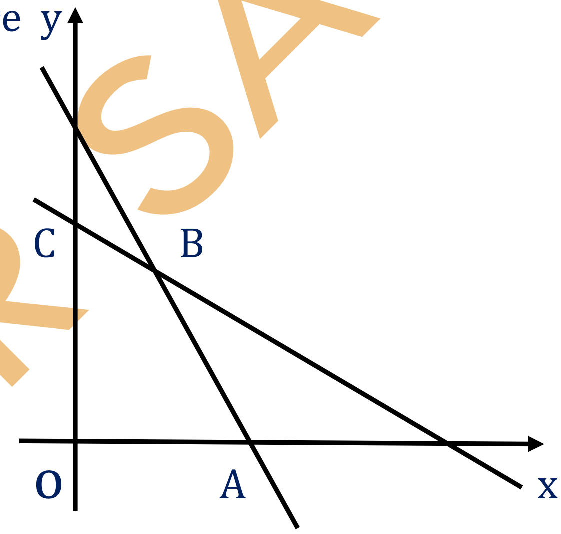
    $$x^2 + y^2 \leq 5,$$
    $$x, y \geq 0.$$
11. Can a LPP have an optimal solution if the feasible region is unbounded. Explain.
12. What are the corner points of the feasible region of a LPP, if the constraints are $3x + 2y \leq 18, x \leq 4, y \leq 6, x, y \geq 0$?

XII. LINEAR PROGRAMMING- SOLUTIONS

1. A set (here in the case of a LPP) is a plane region with the property that the line segment joining any two points in it wholly lies in it.
   Otherwise, it is a non – convex set.

CONVEX SETS *NON-CONVEX SETS*

2. The corner points of the feasible region are
   i) the points of intersections of the graphs of constraints(straight lines) with the co-ordinate axes;
   ii) the points of intersections of the graphs of constraints(straight lines) among themselves.



3. In a linear programming problem (LPP) the objective function (the function which is to be maximised or minimised) should be a linear function of x and y.
   Hence, the LP method cannot be used for the given function.
   [In fact, the given function is a quadratic function in x and y.]
4. In a linear programming problem (LPP) the objective function (the function which is to be maximised or minimised) should be a linear function of x and y.
   Hence, the LP method cannot be used for the given function.
   [In fact, the given function is a rational function in x and y.]
5. False. If two corner points of the feasible region give the same maximum

or minimum of the objective function, then every point on the line segment joining these two points also gives an optimal solution. Therefore, the solution of a LPP, if exists, need not be unique.

6. False. In the case of unbounded region as feasible region, the solution may not exist.

7. Yes. It can be an objective function.

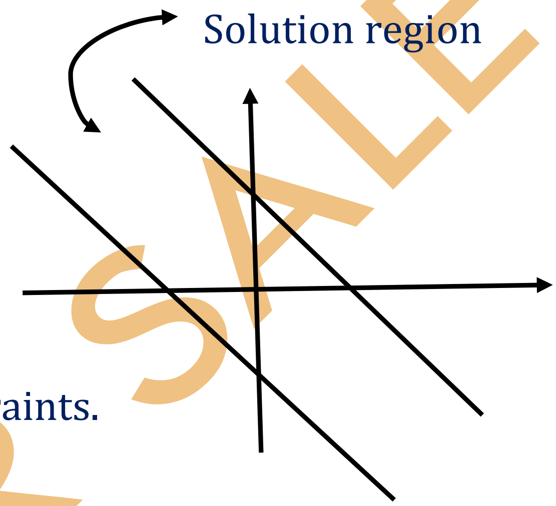
8. The feasible region is unbounded.

   Hence, solution does not exist.

   $|x + y| < 5$

   $\Rightarrow -5 < x + y < 5 \Rightarrow x + y > -5,$

   $x + y < 5$ are the constraints.

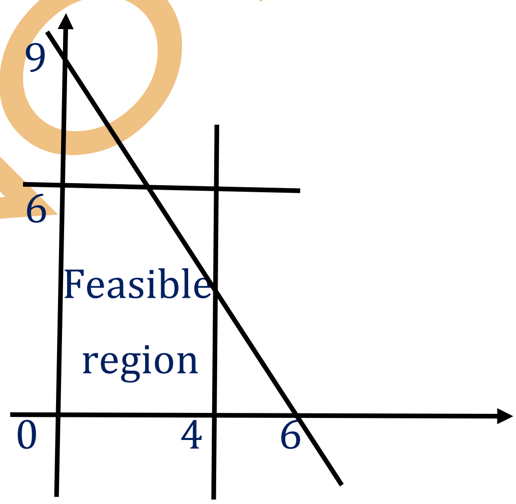


9. The constraints are the same as in Qn 8, together with $x, y \geq 0$. Hence, the feasible region is bounded and solution exists.

10. No, in a LPP the objective function and all the (non-negative) constraints must be linear in the variables.

11. Yes, in case of unbounded region as feasible region also, the function can have a solution.

12.



The corner points of the LPP are (0,0),(4,0), (4,3), (2,6) and (0,6).

END OF CHAPTER 12

XIII. PROBABILITY

VERY SHORT ANSWER TYPE

1. A and B are two events associated with a sample space S, where A and B are independent. What is P(A|B) ?
2. A and B are two events associated with a sample space S,where A and B are independent. What is P(B|A) ?
3. A and B are two events associated with a sample space S,where $A \subset B$. What is P(A|B) ?
4. A and B are two events associated with a sample space S,where $A \subset B$. What is P(B|A) ?
5. A family has 3 children. What is the probability that the youngest one is a girl, given that the family has 2 girls ?.
6. Two coins are tossed simultaneously. What is the probability for getting two heads given that there is atleast one head ?
7. If the events A and B are disjoint partitions of a sample space S, and if F is any other event , then what is P($(A \cup B)$|F) ?.
8. A die is thrown twice and the sum of the numbers is observed to be 8. What is the probability that the second die shows 2 ?,
9. Two coins are tossed once, where the events E and F are E : exactly one head appears, F : atleast one head appears. Find P(E|F).
10. Two unbiased dice have two of their sides painted blue, two sides yellow, one orange and one pink. When the two dice are thrown, what is the probability of getting both dice show the same colour ?
11. A pair of dice is rolled. What is the probability that second dice shows a bigger number than the first, given that the sum is a prime number ?

12. Two cards are drawn at random from a pack of 52 playing cards. What is the probability that they are both aces ?
13. A pair if dice is rolled. What is the probability that the second dice shows a bigger number than the first ?
14. If a die is rolled 4 times, what is the probability that the face 6 comes up atleast once ?
15. A group of 6 men and 6 women is randomly divided into two groups of 6 each. What is the probability that both group will have the same number of men ?
16. A die is rolled once. Find the probability of getting a prime knowing that the outcome is an odd number.
17. If P(A) = 7/13, P(B) = 9/13 and P(A $\cap B$) = 4/13. Find P($\overline{A}$ | B).
18. A die is thrown twice and the sum of numbers appearing is observed to be 6. What is the probability that 4 has appeared atleast once ?

Solve questions 19 – 22 given that :

E and F are two independent events such that P(E) = ¼,P(F) = 1/3.

19. Find P(E $\cap F'$).
20. Find P(E′ $\cap F'$).
21. Find P(E’ $\cap F$).
22. Find P(E′ $\cup F'$).
23. Two persons A and B appear for an interview for two vacancies. The probability for their selection is 1/3 for A and 1/5 for B. Find the probability that none of them will be selected ?
24. A problem in statistics is given to 3 students A,B,C. Their chances of solving it are $\frac{1}{3}, \frac{1}{4}, \frac{1}{5}$. What is the probability the problem is solved ?
25. Two dice are thrown. A is the event of getting a prime on the first die. B is event of getting an even number on the second die. Find P(A $\cap B$).
26. Two dice are thrown. A is the event of getting a prime on the first die.

B is the event of getting an even number on second die. Find P(A|B).

27. Two dice are thrown. A is the event of getting a prime on the first die. B is the event of getting an even number on second die. Find P(B|A).
28. If A and B are two events such that $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$. Find $P(A'|B)$.
29. If A and B are two events such that $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$. Find $P(A|B')$.
30. If A and B are two events such that $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$. Find $P(A'|B')$.
31. If A and B are two events such that $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$. Find $P(B'|A)$.
32. If A and B are two events such that $P(A) = 1/2$, $P(B) = 1/3$ and $P(A \cap B) = 1/4$. Find $P(B|A')$.
33. Two fair dice are thrown simultaneously. A is the event of getting same number on each die and B is the event of getting a sum 10 or more. Are A and B independent ?
34. If A and B are two independent events such that $P(A) = 3/5$, $P(B) = 4/9$. Find $P(A' \cap B')$.
35. If A and B are two independent events such that $P(A) = 3/5$, $P(B) = 4/9$. Find $P(A' \cup B')$.
36. If A and B are two independent events such that $P(A') = 2/5$, $P(B') = 5/9$. Find $P(A \cup B)$.
37. Two cards are drawn from a well shuffled pack of 52 playing cards with replacement. What is the probability that neither card is queen ?.
38. Two cards are drawn from a well shuffled pack of 52 playing cards with replacement. What is the probability that one is a queen card ?

39. a) Two digits are chosen at random without replacement from the set of integers {1,2,3,,4,5,6,7,8}. What is the probability that both digits are greater than 5 ?

b) Two digits are chosen at random with replacement from the set of integers {1,2,3,,4,5,6,7,8}. What is the probability that one of the digits is greater than 5 ?

40. Two roads lead away from a prison. A prisoner escapes from the prison and selects a road at random. If road I is selected, his probability of escaping is 1/8 and for road II, the probability of escaping is 9/10. What is the probability that the prisoner will succeed in escaping ?.

41. Two persons A and B agree for a gamble of tossing 2 coins. A tosses the coin and says he will win Rs.10 if he gets one head and Rs 25 if he gets 2 heads while he loses Rs. 40 if both are tails. If the random variable X is the amount which A can win, write the values of the random variable X and also find the respective probabilities.

42. The probability distribution of a random variable X is as follows :

$X = x$	0	1	2	3	4	5
$P(X = X)$	1/3	¼	k	1/6	$2k^2$	1/4

Find $P(X \geq 3)$.

43. The probability distribution of a random variable X is as follows :

$X = x$	0	1	2	3	4	5
$P(X = X)$	1/3	1/4	k	1/6	$2k^2$	1/4

Find $P(2 < X < 5)$.

44. The probability distribution of a random variable X is as follows :

$X = x$	0	1	2	3	4	5
$P(X = X)$	1/3	1/4	k	1/6	$2k^2$	1/4

Find $P(X < 4)$.

45. The probability distribution of a random variable X is as follows :

X = x	0	1	2	3	4	5
P(X = X)	1/6	5/18	2/9	1/6	1/9	1/8

Find $P(2 < X < 5)$.

46. The probability distribution of a random variable X is as follows :

X = x	0	1	2	3	4	5
P(X = X)	1/6	5/18	2/9	1/6	1/9	1/18

Find $P(|X - 2| \leq 1)$.

47. A box contains 20 oranges out of which 2 are bad. Johnny draws 3 oranges without replacement. If the random variable X denotes the number of bad oranges drawn, what are the values of the random variable ? Find the probability of drawing 0 bad orange.

48. An unbiased die is thrown. Find the probability distribution of the number shown on the die.

49. It is found that a biased coin always shows 4 heads out of 5 tosses. Approximately, in how many sets of 5 tosses of the coin,all tails will occur?

50. Let A, B be two events with P(A) = 0.4, $P(A \cup B) = 0.7$. Let P(B) = p. For what value of p, the events A and B are independent ?

51. Let A, B be two independent events associated with a sample space S and P(A) = 1/3, P(B) = ¼. Find $P(A \cup B)$.

52. Let A, B be two independent events associated with a sample space S and P(A) = 1/3, P(B) = ¼. Find $P((A|A \cup B))$.

53. Let A, B be two independent events associated with a sample space S and P(A) = 1/3, P(B) = ¼. Find $P((B|A \cup B))$.

XIII. PROBABILITY

[CASE STUDY BASED M C QS]

I. A total of 500 married working couples were questioned about their annual salaries, with the following information resulting :
Now, answer the following questions :

	HUSBAND	
WIFE	< Rs. 25,000	> Rs. 25,000
< Rs. 25,000	212	198
> Rs. 25,000	36	54

One of the couples is randomly chosen.

1. The probability that the husband earns less than Rs. 25,000 is

   a) 62/125 b) 15/42 c) 9/53 d) 45/124

2. The probability that the wife earns more than Rs. 25,000 is

   a) 9/50 b) 15/42 c) 9/53 d) 45/124

3. The conditional probability of wife earning more than Rs. 25,000 given that the husband also earns more than Rs. 25,000 is

   a) 3/14 b) 15/42 c) 9/53 d) 45/124

4. The probability that both earn less than Rs. 25,000 is

   a) 53/125 b) 45/124 c) 9/53 d) 15/42

5. The conditional probability that the wife earns more than Rs. 25,000 given that the husband earns less than Rs. 25,000 is

   a) 9/62 b) 45/124 c)9/53 d) 15/42.

II. It is found that color blindness is inheritable. Since the responsible gene is sex-linked, color blindness occurs more frequently in males than in females. In a large human population the occurrence of red-green color blindness was counted and the information is provided below.

Assume that the population consists of 20,000 people.

	Male	Female	Total
Color-blind	4.23 %	0. 65%	4.88 %
Normal	48.48 %	46. 64 %	95.12 %
Total	52.71 %	47.29 %	100.00 %

A person is chosen at random to find if he/she is colorblind.
Now, answer the following questions :

1. What is the conditional probability that the chosen person is colorblind given that the person is a male ?

   a) 0.0423 b) 0.4848 c) 0.5271 d) 0.0803

2. What is the probability that the chosen person is a normal male ?

   a) 0.4848 b) 0.5271 c) 0.0423 d) 0.9197

3. What is the conditional probability that chosen person is colorblind given that the person is a female ?

   a) 0.4729 b) 0.0137 c) 0.4664 d) 0.0065

4. In the population, the number of males having colorblindness is

   a) 976 b) 488 c) 846 d) 423

5. In the population, the total number of colorblindness is

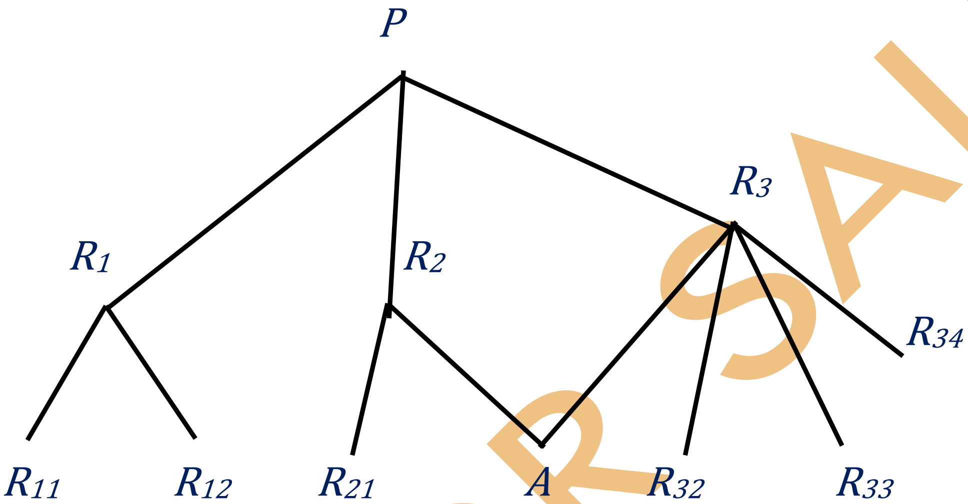
   a) 846 b) 976 c) 488 d) 19024

III. Four roads lead away from a jail. A prisoner has escaped from the jail and selects a road at random. If road I is selected, the probability of escaping is 1/8; if road II is selected, the probability of success is 1/6; if road III is selected, the probability of escaping is ¼; and if road IV is selected, the probability of success is 9/10.

Now, answer the following questions :

1. What is the probability that the prisoner will succeed in escaping ?
   a) 173/240 b) 108/173 c) 173/480 d) 15/173
2. If the prisoner succeeds, what is the probability that the prisoner escaped by using road IV ?
   a) 15/173 b) 108/173 c) 173/240 d) 9/40
3. If the prisoner succeeds, what is the probability that the prisoner escaped by using road I ?
   a) 173/240 b) 173/480 c) 108/173 d) 15/173
4. What is the probability that the prisoner will not succeed in escaping ?
   a) 9/40 b) 108/173 c) 307/480 d) 173/240
5. Which road gives the prisoner the maximum likelihood of escaping ?
   a) road I b) road II c) road III d) road IV

IV. Consider a bicyclist who leaves a point P, choosing one of the roads PR_1, PR_2, PR_3 at random. At each subsequent crossroad she again chooses a road at random.



Now, answer the following questions :

1. What is the probability that she will arrive at point A ?

   a) ¼ b) ½ c) 2/3 d) 1/3

2. What is conditional probability that she will arrive at A via road PR_3?

   a) 3/7 b) 1/3 c) ¼ d) 2/3

3. What is the probability that she not will arrive at point A ?

   a) 1/3 b) ¼ c) 2/3 d) ¾

4. What is conditional probability that she will arrive at A via road PR_2?

   a) ½ b) 1/3 c) 2/3 d) ¾

5. What is the probability that she will not arrive at A given that she chose PR_3?

   a) 2/3 b) ¼ c) ¾ d) 5/12.

V. Two machines M and N manufacture a component. The probability that The component is of an acceptable standard is 0.95 when produced by machine M and 0.85 when manufactured by machine N. Machine M supplies 60% of components, machine N supplies 40 % components. A component is picked at random.

1) What is probability that the component is of an acceptable standard?

a) 0.9000 b) 0.9100 c) 0.0900 d) 0.9500

2) What is the probability that the component is of an acceptable Standard and is made by machine M ?

a) 0.5700 b) 0.9100 c) 0.3700 d) 0.8075

3) What is the probability that the component was made by M ?

a) 0.5700 b) 0.6000 c) 0.6700 d) 0.3800

4) What is the probability that the component was made by machine M given that it is of an acceptable standard ?

a) 0.6264 b) 0.5700 c) 0.6000 d) 0.8075

5) The component is not of an acceptable standard. What is the probability that it was made by machine N ?

a) 0.6667 b) 0.1500 c) 0.3400 d) 0.4300

VI. On average, Mr. Bean travels to work by bus 10% of the time, train 60% of the time and car the rest of the time. IF he travels by bus there is a probability of 0.1 that he will be late. If he travels by train, there is probability 0.2 that he will be late and if he travels by car, there is probability of 0.3 that he is late.

Now, answer the following questions :

1. If he is observed on a given particular day, what is the probability that he is late for work ?

   a) $\frac{21}{100}$ b) $\frac{22}{100}$ c) $\frac{27}{100}$ d) $\frac{7}{100}$

2. If it is known that he is late for work on a given day, what is the probability that he came by bus ?

   a) $\frac{1}{22}$ b) $\frac{12}{22}$ c) $\frac{9}{22}$ d) $\frac{7}{22}$

3. What percentage of days he is likely to have arrived for work on time?

   a) 79%   b) 73%   c) 91%   d) 78%

4. Which mode of transport he should prefer to be less likely late ?

   a) train   b) bus   c) car   d) bus or car

5. During a given year, how many days he is expected to come late for work ? (approximately)

   a) 80   b) 77   c) 99   d) 33

VII. A scientist wants to know whether there is any dependence between color blindness and deafness in human males. Assume that he is given the following probabilities:

	Deaf	Not deaf	Total
Color-blind	0.0004	0.0796	0.0800
Not color-blind	0.0046	0.9154	0.9200
Total	0.0050	0.9950	1.0000

Now, answer the following questions :

1. $P(D|C) =$

   a) 0.0050 b) 0.0004 c) 0.0800 d) 0.9950

2. $P(\bar{D}|C) =$

   a) 0.9950 b) 0.0796 c) 0.0800 d) 0.0004

3. $P(C|D) =$

   a) 0.0800 b) 0.0050 c) 0.0004 d) 0.9200

4. $P(\bar{C}|D) =$

   a) 0.9200 b) 0.0046 c) 0.0004 d) 0.0008

5. Which of the following is true ?

   a) $P(D|C) < P(D)$ b) $P(D|C) > P(D)$

   c) D and C are independent d) none of these.

CHAPTER XIII - SOLUTIONS

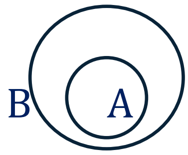
1. A and B are independent. So, $P(A \cap B) = P(A) \cdot P(B)$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A).$$

2. A and B are independent. So, $P(A \cap B) = P(A) \cdot P(B)$.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = P(B).$$

3. $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)}{P(B)}$.



← Qn.3, Qn.4

4. $P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1.$

5. The family has 3 children, say BBB,BBG,BGB,GBB,BGG,GBG,GGB,GGG.

Let E:youngest is a girl, F:The family has 2 girls.$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{2/8}{3/8} = \frac{2}{3}$.

6. The outcomes are HH,HT,TH,TT. Let E: two heads,F:atleast one head

$E \cap F = \{HH\}$. $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/4}{3/4} = \frac{1}{3}$.

7. $P[(A \cup B)|F] = \frac{P[(A \cup B) \cap F]}{P(F)} = \frac{P(F)}{P(F)} = 1.$

8. Sample space, S = {(1,1) ... (6,6)}. 36 outcomes.

E: second die shows 2 → 12,22,32,42,52,62, F: sum is 8 → 26,35,44,53,62,

the event $E \cap F = \{62\}$. $P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/36}{5/36} = \frac{1}{5}$.

9. Sample space, S = {HH,HT,TH,TT}

E: exactly one head , F : atleast one head , $E \cap F$: HT, TH

: HT,TH : HT,TH,HH

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{2/4}{3/4} = \frac{2}{3}.$$

10. It is given that in each of the two dice, two sides painted blue, two sides red, one yellow and one pink. [In normal dice with numbers, we'll have the outcomes (1,1),(1,2)...(6,6)].

But, here the outcomes are RR,RB,RY,PY etc.

Now, P(Both dice show the same color) = P(RR or BB or YY or PP)

= P(RR) + P(BB) + P(YY) + P(PP) , the events are mutually exclusive.

= P(R)P(R) + P(B)P(B) + P(Y)P(Y) + P(P)P(P) , because independent

$= \frac{2}{6}\frac{2}{6} + \frac{2}{6}\frac{2}{6} + \frac{1}{6}\frac{1}{6} + \frac{1}{6}\frac{1}{6} = \frac{5}{18}$.

11. E : second bigger than first, F: sum is prime

: 12,13,14,15,16 23,24,25,26 34,35,36,45,46,56	: 11,12,14,16,21,23,25, 32,34, 41,43,52,56,61,65 $E \cap F$: 14,16,23,25,34,56	P(E|F) $= \frac{P(E \cap F)}{P(F)}$ $= \frac{6/36}{15/36} = \frac{2}{5}$.

12. P(both aces) = ${}_4C_2/{}_{52}C_2$ = 1/221.

13. **PLEASE TRY YOURSELF.** [ANS : 5/12].

14. P(atleast one 6) = 1 – P(no six) = 1- $\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6}\cdot\frac{5}{6} = 1 - \left(\frac{5}{6}\right)^4$.

15. $\frac{6C_3 \times 6C_3}{12C_3} = \frac{100}{231}$.

16. Sample space, S = {1,2,3,4,5,6}. Let E : a prime ; 2,3,5

F : outcome is odd ; 1,3,5. P(E|F) = $\frac{2/6}{3/6} = \frac{2}{3}$.

17. $P(\bar{A}|B) = \frac{P(\bar{A} \cap B)}{P(B)} = \frac{P(B) - p(A \cap B)}{P(B)} = 1 - \frac{4/13}{9/13} = 5/9$.

18. E: atleast one 4

:14,24,34,44,54,64

41,42,43,45,46

F: sum 6

: 15,24,33,42,51

$E \cap F$: 24,42

$P(E|F) = \frac{P(E\cap F)}{P(F)}$

$= \frac{2/36}{5/36} = \frac{2}{5}.$

19. P(E) = ¼, P(F) = 1/3 ; [P(F') = 1 – P(F) = 1 – 1/3 = 2/3]

E and F are independen ⇒ E and F' also independent.

$P(E \cap F') = P(E).P(F') = \frac{1}{4}\cdot\frac{2}{3} = \frac{1}{6}.$

20. P(E) = ¼, P(F) = 1/3 ;

E and F are independent ⇒ E' and F' also independent.

[P(E') = 1 – P(E) = 1 – 1/4 = 3/4.] & [P(F') = 1 – P(F) = 2/3.]

Therefore, $P(E' \cap F') = P(E').P(F') = \frac{3}{4}\cdot\frac{2}{3} = \frac{1}{2}.$

21. P(E) = ¼, P(F) = 1/3 ; [P(E') = 1 – P(E) = 1 – 1/4 = 3/4.]

E and F are independent ⇒ E' and F also independent.

Therefore, $P(E \cap F') = P(E).P(F') = \frac{1}{4}\cdot\frac{2}{3} = \frac{1}{6}.$

22. P(E) = ¼, P(F) = 1/3. So, $P(E' \cup F') = P(E \cap F)' = 1 - P(E \cap F)$

$= 1 - P(E).P(F) = 1 - \frac{1}{4}\cdot\frac{1}{3} = \frac{11}{12}.$

23. Let A: the event that A gets selected ;

B : the event that B gets selected. P(A) = 1/3 , P(B) = 1/5.

P(none of them gets selected) = P(A' and B') = $P(A' \cap B')$

$= 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - [P(A) + P(B) - P(A)\cdot P(B)]$ A & B are indep.

$= 1 - [1/3 + 1/5 - 1/3\cdot 1/5] = 8/15.$

24. P(A) = 1/3, P(**B**) = 1/4, P(C) = 1/5.

Now, P(A') = 2/3, P(**B'**) = 3/4, P(C') = 4/5

P(the problem is solved) = 1- P(the problem is not solved)

$= P(A' \cap B' \cap C') = P(A') \cap P(B') \cap P(C') = 2/5.$

25. Sample space, $S = \{ (1,1),(1,2), \dots , (6,6)\}$; $n(S) = 36$

The event A : getting prime on the first die

: (2,1),…,(2,6),(3,1)…,(3,6),(5,1),…,(5,6); n(A) = 18

B : even number on the second die

: (1,2),(1,4),(1,6),(2,2),(2,4),(2,6),…,(6,6); n(B) = 12

$P(A \cap B) = 3/36 = 1/12.$

26. A : prime number on the first die

: 21,22,23,24,25,26,
31,32,33,34,35,36,
51,52,53,54,55,56.

B : even number on second die

: 12,14,16,22,24,26,
32,34,36,42,44,46,
52,54,56,62,64,66.

$A \cap B$: 22,24,26,32,34,36,52,54,56.

$P(A) = 18/36 = 1/2$; $P(B) = 18/36 = 1/2$; $P(A \cap B) = 9/36 = ¼$.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{9/36}{18/36} = \frac{1}{2}.$$

27. Refer Qn 26. $P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{9/36}{18/36} = \frac{1}{2}.$

For questions 28 – 32, P(A) = ½, P(B) = 1/3, P(A ∩ B) = ¼.

28. $P(A'|B) = \frac{P(A' \cap B)}{P(B)} = \frac{P(B)-p(A\cap B)}{P(B)} = 1 - \frac{1/4}{1/3} = \frac{1}{4}.$

29. $P(A|B') = \frac{P(A \cap B')}{P(B')} = \frac{P(A)-p(A\cap B)}{1-P(B)} = \frac{1/2 - 1/4}{1-1/3} = \frac{1/4}{2/3} = \frac{3}{8}.$

30. $P(A'|B') = \frac{P(A' \cap B')}{P(B')} = \frac{p(A \cup B)'}{1-P(B)} = \frac{1 - P(A\cup B)}{1-P(B)} = \frac{1 - [P(A) + P(B) - P(A \cap B)]}{1-P(B)}$

$$= \frac{1 - [1/2 + 1/3 - 1/4]}{1 - 1/3} = \frac{1 - 7/12}{2/3} = \frac{5}{8}.$$

31. $P(B'|A) = \frac{P(B' \cap A)}{P(A)} = \frac{P(A)-p(A\cap B)}{P(A)} = 1 - \frac{1/4}{1/2} = \frac{1}{2}.$

32. $P(B|A') = \frac{P(B \cap A')}{P(A')} = \frac{P(B) - p(A \cap B)}{1 - P(A)} = \frac{1/3 - {}^{1}/_{4}}{1 - {}^{1}/_{2}} = \frac{1}{6}$.

33. Sample space, $S = \{ (1,1),(1,2), \dots , (6,6)\}$; $n(S) = 36$.

A : same number on both dice : (1,1),(2,2),(3,3),(4,4),(5,5),(6,6)

B : a sum 10 or more : (4,6),(6,4),(5,5),(5,6),(6,5),(6,6) ;

$A \cap B = (5,5),(6,6)$. $P(A) = 1/6$; $P(B) = 1/6$; $P(A \cap B) = 2/36$.

$P(A \cap B) \neq P(A) \cdot P(B)$. So, A and B are not independent.

34. Given, $P(A) = 3/5$, $P(B) = 4/9$. Also, given A and B are independent,

So $P(A \cap B) = P(A) \cdot P(B) = \frac{3}{5} \cdot \frac{4}{9} = \frac{4}{15}$.

Therefore, $P(A' \cap B') = P(A \cup B)' = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B) - P(A \cap B)]$

$= 1 - [3/5 + 4/9 - 4/15] = 1 - 7/9 = 2/9$.

__OR__, since A and B are independent, A' and B' are also independent.

Therefore, $P(A' \cap B') = P(A').P(B') = [1 - P(A)][1 - P(B)] = \frac{2}{5} \cdot \frac{5}{9} = 2/9$.

35. Refer Qn 34. $P(A' \cup B') = P(A \cap B)' = 1 - P(A \cap B)$

$= 1 - P(A).P(B) = 1 - \frac{3}{5} \cdot \frac{4}{9} = \frac{11}{15}$.

36. Given, A and B are independent and $P(A') = 2/5, P(B') = 5/9$.

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(A) + P(B) - P(A) \cdot P(B)$, $P(A \cap B) = P(A) \cdot P(B)$, since A and B are independent.

$= 3/5 + 4/9 - 3/5.4/9$

$= 7/9$.

37, Two cards are drawn with replacement(that is one after the other).

Let A denote the event no queen on the first draw and B denote the event no queen on the second draw. A and B are independent.

Therefore, P(no queen) = P(A and B) = P(AB)

$$= P(A)P(B) = \frac{48}{52}\cdot\frac{48}{52} = \frac{144}{169}.$$

38. Two cards are drawn with replacement(that is one after the other).

Let A denote the event of getting a queen card, then A' denote the event of not getting a queen card. This may happen this way.

Draw	First	Second
Card	Queen	No queen
OR	No queen	Queen

Now, P(One queen card) = P(AA' or A'A) = P(AA') + P(A'A)

$$= \frac{4}{52}\cdot\frac{48}{52} + \frac{48}{52}\cdot\frac{4}{52} = \frac{24}{169}.$$

since only one of AA' and A'A can happen, they are exclusive.

For two mutually exclusive events A and B, P(A U B) = P(A) + P(B).

Because in case of ME events $A \cap B$ is empty and so $P(A \cap B) = 0$.

39. a) The two digits are chosen without replacement .(i.e., one after the other without putting back the first drawn digit before the second draw).

A: first digit > 5 B : second digit > 5

The digits greater than 5 are 6,7,8.

P(Both digits greater than 5) $= P(A \cap B) = P(A).P(B|A) = \frac{3}{8}\cdot\frac{2}{7} = \frac{3}{28}$.

b) P(A or B) = P((A and not B) OR (not A and B))

= P(A and not B) + P(not A and B)

$$= \frac{3}{8}\cdot\frac{5}{8} + \frac{5}{8}\cdot\frac{3}{8} = \frac{15}{32}$$

40. Let E be the event that the prisoner escapes and R_1 and R_2 be the be the events of choosing road I and road II respectively.

$P(R_1) = \frac{1}{2}$, $P(R_2) = \frac{1}{2}$. $P(A|R_1) = 1/8$, $P(A|R_2) = 9/10$.

Therefore, $P(E) = P(R_1).\,P(A|R_1) + P(R_2).\,P(A|R_2) = \frac{1}{2}\cdot\frac{1}{8} + \frac{1}{2}\cdot\frac{9}{10} = \frac{41}{80}$.

41. The random variable is X : the amount that X can win

: (Rs.) 10, 25, - 40.

The outcomes are HH,HT,TH,TT.

P(for winning Rs.10) = P(one head) = 2/4 = ½.

P(for winning Rs. 25) = P(2 heads) = ¼,

P(for winning Rs.(- 40)) = P(losing Rs. 40) = P(both tails) = ¼.

42. Total probability = Sum of all probabilities = 1.

Therefore, $1/3 + ¼ + k + 1/6 + 2k^2 + 1/4 = 1$; $k + 2k^2 = 0$. k = 0,

k cannot be -1/2 as probability cannot be negative.

$P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5)$

$= 1/6 + 0 + 1/4 = 5/12$.

43. Total probability = Sum of all probabilities = 1.

Therefore, $1/3 + ¼ + k + 1/6 + 2k^2 + 1/4 = 1$

$k + 2k^2 = 0$. k = 0 , k cannot be -1/2 as probability cannot be negative.

$P(2 < X < 5) = P(X = 3) + P(X = 4) = 1/6 + 0 = 1/6$.

44. Total probability = Sum of all probabilities = 1.

Therefore, $1/3 + ¼ + k + 1/6 + 2k^2 + 1/4 = 1$

$k + 2k^2 = 0$. k = 0 , k cannot be -1/2 as probability cannot be negative.

$P(X < 4) = P(X = 0)+P(X = 1) + P(X = 2) + P(X = 3)$

$= 1/3 + ¼ + 0 + 1/6 = ¾$.

45. $P(2 < X < 5) = P(X = 3) + P(X = 4) = 1/6 + 1/9 = 5/18$.

46. $|X - 2| \leq 1 \Rightarrow -1 \leq X - 2 \leq 1 \Rightarrow 1 \leq X \leq 3$.

Therfore, $P(|X - 2| \leq 1) = P(1 \leq X \leq 3) = P(X = 1) + P(X = 2) + P(X = 3)$

$= 5/18 + 2/9 + 1/6 = 2/3$.

47. No. of good oranges = 18, Bad oranges = 2, Total = 20.

Three oranges are drawn without replacement(one after the other).

Let G denote a good orange, B denote a bad orange.

The values of the RV X : 0,1,2

P(0 bad orange) = P(all three good) = P(GGG) = $\frac{18}{20} \cdot \frac{17}{19} \cdot \frac{16}{18} = \frac{68}{95}$.

48. If a die is thrown, the outcomes are 1,2,3,4,5,6.

The probabilities are $\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}$. The probability distribution is

X = x	1	2	3	4	5	6
P(X = x)	1/6	1/6	1/6	1/6	1/6	1/6

Where, each $p_i \geq 0$ *and* $\sum p_i = 1$.

49. In every set of 5 tosses, a biased coin, shows 4 heads.

So, P(all tails) = P(TTTTT) = $\frac{1}{5}\frac{1}{5}\frac{1}{5}\frac{1}{5}\frac{1}{5} = \frac{1}{3025}$.

∴ In 3025 sets of tosses, once all tails can be expected to occur.

50. Given, P(A) = 0.4, P(AUB) = 0.7, P(B) = p.

Since, A and B are independent, $P(A \cap B) = P(A) \cdot P(B)$.

Now, P(A U B) = P(A) + P(B) – $P(A \cap B)$ = P(A) + P(B) - $P(A) \cdot P(B)$

⇒ 0.7 = 0.4 + p – 0.4p, Solving, p = ½.

51. Given, P(A) = 1/3, P(B) = 1/4, P(B)

Since, A and B are independent, $P(A \cap B) = P(A) \cdot P(B)$.

Now, P(A U B) = P(A) + P(B) – $P(A \cap B)$ = P(A) + P(B) - $P(A) \cdot P(B)$

= 1/3 + ¼ - 1/3 x ¼ = ½.

52. Given, P(A) = 1/3, P(B) = 1/4, P(B)

Since, A and B are independent, $P(A \cap B) = P(A) \cdot P(B)$.

$$P(A|AUB) = \frac{P[A \cap (A \cup B)]}{P(A \cup B)} = \frac{P(A)}{P(AUB)} = \frac{1/3}{1/3+1/4-1/12} = 2/3.$$

53. Given, $P(A) = 1/3$, $P(B) = 1/4$, $P(B)$

Since, A and B are independent, $P(A \cap B) = P(A) \cdot P(B)$.

$$P(B|AUB) = \frac{P[B \cap (A \cup B)]}{P(A \cup B)} = \frac{P(B)}{P(AUB)} = \frac{1/4}{1/3+1/4-1/12} = 1/2.$$

CHAPTER XIII – SOLUTIONS TO CASE STUDIES

I. 1. P(husband earns less than Rs.25,000) $= \frac{212+36}{500} = \frac{62}{125}$.

2. P(wife earns more than Rs.25,000) $= \frac{54+36}{500} = \frac{9}{50}$.

3. P(wife earns > 25000 | husband earns > 25000)

$$= \frac{P(both > 25000)}{P(husband > 25000)} = \frac{54/500}{252/500} = \frac{3}{14}.$$

4. P(both earn less than 25000) $= \frac{212}{500}$.

5. P(wife earns > 25000 | husband earns < 25000) $= \frac{36/500}{248/500} = \frac{9}{62}$.

II. 1. The conditional probability

P(Chosen person is colorblind|the person is male)

$$= \frac{4.23/100}{52.71/100} = 0.0803.$$

2. P(chosen person is a normal male) $= \frac{48.48/100}{52.71/100} = 0.9197$.

3. The conditional probability P(Chosen person is colorblind |

the person is female) = $\frac{0.65/100}{47.29/100} = 0.0137.$

4. The number of males having color blindness = $\frac{4.23}{100}\ x\ 20000 = 846.$

5. The total number of persons with color blindness = 976.

III. Let E denote the event that the prisoner escapes and let R_1, R_2, R_3 and R_4 denote events that the prisoner chooses roadI,road II, road III,road IV respectively. All the roads can be equally chosen.

Therefore, $P(R_1) = P(R_2) = P(R_3) = P(R_4) = \frac{1}{4}$.

Also, the respective probabilities of escaping using different roads is :

$P(E|R_1) = 1/8$, $P(E|R_2) = 1/6$, $P(E|R_3) = ¼$, $P(E|R_4) = 9/10$.

1. $P(E) = P(R_1)\ P(E|R_1) + P(R_2)\ P(E|R_2) + P(R_3)\ P(E|R_3) + P(R_4)\ P(E|R_4)$

$= 1/4x1/8 + 1/4x1/6+1/4x1/4+1/4x9/10 = 173/480.$

2. P(the prisoner succeeds using road IV) =

$P(R_4 \mid E) = \frac{P(R_4)P(E|R_4)}{P(E)} = \frac{1/4X9/10}{173/480} = \frac{108}{173}.$

3. P(the prisoner succeeds using road I) = 15/173.

4. P(the prisoner will not succeed in escaping) = P(E')= 1–P(E) = $\frac{307}{480}$.

5. Chance of escaping using road IV is 0.9, higher than any other road.

IV. Let A be the event that she arrives at A and let PR_1, PR_2, PR_3 denote the events of choosing roads R_1, R_2, R_3 respectively.

All the roads can be equally chosen. So, $P(R_1) = P(R_2) = P(R_3) = \frac{1}{3}$

Also, respective probabilities of arriving at A using different roads is :

$P(A|R_1) = 0$, $P(A|R_2) = 1/2$, $P(A|R_3) = 1/4$.

1. $P(A) = P(R_1)\,P(A|R_1) + P(R_2)\,P(A|R_2) + P(R_3)\,P(A|R_3)$

   $= 1/3 \times 0 + 1/3 \times 1/2 + 1/3 \times 1/4 = \frac{1}{4}$.

2. P(she arrives using road PR_3) $= P(R_3|A) = \frac{P(R_3)P(A|R_3)}{P(A)} = \frac{1/3 X 1/4}{1/4} = \frac{1}{3}$.

3. P(she will not arrive at A) $= P(A') = 1 - P(A) = \frac{3}{4}$.

4. P(she arrives using road PR_2) $= P(R_2|A) = \frac{P(R_2)P(A|R_2)}{P(A)} = \frac{1/3 X 1/2}{1/4} = \frac{2}{3}$.

5. P(not arriving at A given that PR_3 is chosen) $= P(A'|R_3) = \frac{3}{4}$.

V. Define the events as :

M : the component is manufactured by machine M

N : the component is manufactured by machine N

C : the component is of an acceptable standard

Consider 1000 components.

Then, 600 are manufactured by machine M and

400 are manufactured by machine N.

Out of the 600 manufactured by M, 95% acceptable.,

that is $\frac{95}{100} X\ 600 = 570$ are acceptable.

Out of the 400 manufactured by N, 85% acceptable.,

that is $\frac{85}{100} X\ 400 = 340$ are acceptable.

Therefore, a total of $570 + 340 = 910$ are acceptable out of 1000.

1. $P(C) =$ P(the component is acceptable) $= \frac{910}{1000} = 0.9100$

2. P(the component is acceptable and made by M) =

   $P(M \cap C) = \frac{570}{1000} = 0.5700$

3. P(the component was made by M) $= \frac{60}{100} = 0.6000$

4. P(component was made by M given that it is acceptable) =

$$P(M|C) = \frac{P(M\cap C)}{P(C)} = \frac{570/1000}{910/1000} = 0.6264$$

5. P(component was made by N given that it is not acceptable) =

$$P(N|C') = \frac{P(N\cap C')}{P(C')} = \frac{60/1000}{90/1000} = 0.6667$$

VI. 1. $\frac{22}{100}$, [Use theorem of total probability]

2. $\frac{1}{22}$, [Calculate P(B|L)] BAYES' THEOREM

3. 100 % – 22 % = 78 %

4. Bus [Calculate P(B|L), P(T|L), P(C|L) and compare which is least]

5. 80 [Multiply 0.22 and 365.25 days]

VII. 1. $P(D|C) = \frac{P(D\cap C)}{P(C)} = \frac{0.0004}{0.0800} = 0.0050.$

2. $P(\bar{D}|C) = \frac{P(\bar{D}\cap C)}{P(C)} = \frac{0.0796}{0.0800} = 0.9950.$

3. $P(C|D) = \frac{P(C\cap D)}{P(D)} = \frac{0.0004}{0.0050} = 0.0800.$

4. $P(\bar{C}|D) = \frac{P(\bar{C}\cap D)}{P(D)} = \frac{0.0046}{0.0050} = 0.9200.$

5. D and C are independent.

END OF CHAPTER 13

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