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SAMPLE PAPER TEST – 02 (2020-21)
(SAMPLE ANSWER)

SUBJECT: MATHEMATICS **MAX. MARKS : 80**
CLASS : XII **DURATION : 3 HRS**

General Instruction:
1. This question paper contains two **parts A and B**. Each part is compulsory. Part A carries **24** marks and Part B carries **56** marks
2. **Part-A** has Objective Type Questions and **Part -B** has Descriptive Type Questions
3. Both Part A and Part B have choices.

Part – A:
1. It consists of two sections**- I and II**.
2. Section **I** comprises of 16 very short answer type questions.
3. Section **II** contains **2** case studies. Each case study comprises of 5 case-based MCQs. An examinee is to attempt **any 4 out of 5 MCQs**.

Part – B:
1. It consists of three sections**- III, IV and V.**
2. Section **III** comprises of 10 questions of **2 marks** each**.**
3. Section **IV** comprises of 7 questions of **3 marks** each**.**
4. Section **V** comprises of 3 questions of **5 marks** each**.**

PART - A
SECTION-I
Questions 1 to 16 carry 1 mark each.

1. Check whether f(x) = tanx is one-one or not on R.

We have, $f(x) = \tan x$

Since, $f(0) = \tan 0 = 0$

and $f(\pi) = \tan \pi = 0$

$\therefore \quad f(0) = f(\pi)$

But $0 \neq \pi$

So, $f(x) = \tan x$ is not one-one.

2. Find the domain of f(x) = $\sin^{-1}(-x^2)$.

The domain of $\sin^{-1} x$ is $[-1, 1]$.

Therefore, $f(x) = \sin^{-1}(-x^2)$ is defined for all x satisfying.

$$-1 \leq -x^2 \leq 1 \quad \Rightarrow \quad 1 \geq x^2 \geq -1$$

$$\Rightarrow \quad 0 \leq x^2 \leq 1 \;\Rightarrow x^2 \leq 1 \Rightarrow x^2 - 1 \leq 0$$

$\Rightarrow \quad (x-1)(x+1) \leq 0 \Rightarrow -1 \leq x \leq 1$

Hence, the domain of $f(x)$ is $[-1, 1]$.

3. If A is a matrix of order 3 x 3 such that |A| = 4, find $|A^{-1}|$.

We have, $|A| = 4$

Now, $|A^{-1}| = \dfrac{1}{|A|} = \dfrac{1}{4}$

4. Find the value of $\cot\left(\frac{\pi}{3}-2\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)\right)$

$$\cot\left(\frac{\pi}{3}-2\cot^{-1}\frac{1}{\sqrt{3}}\right)=\cot\left(\frac{\pi}{3}-2\times\frac{\pi}{3}\right)=\cot\left(\frac{\pi}{3}-\frac{2\pi}{3}\right)$$

$$=\cot\left(-\frac{\pi}{3}\right)=-\cot\frac{\pi}{3}=-\frac{1}{\sqrt{3}}$$

5. If for the matrix A, $A^3 = I$, then find the value of A^{-1}.

We have, $A^3 = I$

$$\Rightarrow A^{-1}A^3 = A^{-1}I$$

$$\Rightarrow A^2 = A^{-1} \quad [\because A^{-1}A = I, A\cdot I = A]$$

$$\therefore A^{-1} = A^2$$

6. If 2A + B + X = 0, where $A=\begin{bmatrix}-1 & 2\\ 3 & 4\end{bmatrix}$ and $B=\begin{bmatrix}3 & -2\\ 1 & 5\end{bmatrix}$, then find the value of matrix X.

We have, $2A + B + X = 0$

$$\Rightarrow X = -(2A + B)$$

Now, $2A+B=2\begin{bmatrix}-1 & 2\\ 3 & 4\end{bmatrix}+\begin{bmatrix}3 & -2\\ 1 & 5\end{bmatrix}$

$$=\begin{bmatrix}-2 & 4\\ 6 & 8\end{bmatrix}+\begin{bmatrix}3 & -2\\ 1 & 5\end{bmatrix}=\begin{bmatrix}-2+3 & 4-2\\ 6+1 & 8+5\end{bmatrix}=\begin{bmatrix}1 & 2\\ 7 & 13\end{bmatrix}$$

$$\therefore X=-(2A+B)=\begin{bmatrix}-1 & -2\\ -7 & -13\end{bmatrix}$$

7. Find the equation of the tangent to the curve $y = x^2 + 4x + 1$ at the point where x = 3.

When $x = 3$, we have $y = (3^2 + 4\times 3 + 1) = 22$

So, the point of contact is (3, 22).

Now, $y = x^2 + 4x + 1$

$$\frac{dy}{dx} = 2x + 4$$

$$\therefore \left(\frac{dy}{dx}\right)_{(3,\ 22)} = 2\times 3 + 4 = 10$$

∴ Equation of tangent at (3, 22)

$$y - 22 = 10(x-3) \Rightarrow y - 22 = 10x - 30$$

$$\Rightarrow 10x - y = 8$$

8. If $y = 7x^3 - 4x^2$, find $\frac{dy}{dx}$.

We have, $y = 7x^3 - 4x^2$

$$\therefore \frac{dy}{dx} = 7\times 3x^2 - 4\times 2x$$

$$= 21x^2 - 8x.$$

9. Evaluate: $\int\frac{dx}{x + x\log x}$

Let $I = \int \frac{1}{x + x\log x}dx = \int \frac{1}{x(1+\log x)}dx$

Let $1 + \log x = t \Rightarrow \frac{1}{x}dx = dt$

$\therefore \quad I = \int \frac{dt}{t} = \log t + C$

$= \log(1 + \log x) + C$

10. If A and B are events such that P(A) = 0.3, P(B) = 0.6 and P(B/A) = 0.5, find P(A/B).

We have, $P(A) = 0.3, P(B) = 0.6$ and $P(B/A) = 0.5$

$\therefore P(A \cap B) = P(A)\,P(B/A) = 0.3 \times 0.5 = 0.15$

$\therefore \quad P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{0.15}{0.6} = \frac{1}{4}$

11. An urn contains 10 black and 5 white balls. Two balls are drawn from the urn one after the other without replacement then find the probability that both drawn balls are black.

Let E and F denote respectively the events first and second ball drawn are black.

$\therefore \quad P(E) = \frac{10}{15},\ P\left(\frac{F}{E}\right) = \frac{9}{14}$

$\therefore$ Required probability, $P(E \cap F) = P(E) \times P\left(\frac{F}{E}\right) = \frac{10}{15} \times \frac{9}{14} = \frac{3}{7}$

12. If $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$, then find the value of $(\vec{a} + 3\vec{b}).(2\vec{a} - \vec{b})$.

We have, $\vec{a} = \hat{i} + \hat{j} + 2\hat{k}$ and $\vec{b} = 3\hat{i} + 2\hat{j} - \hat{k}$

$\therefore \quad \vec{a} + 3\vec{b} = (\hat{i} + \hat{j} + 2\hat{k}) + 3(3\hat{i} + 2\hat{j} - \hat{k}) = 10\hat{i} + 7\hat{j} - \hat{k}$

and $2\vec{a} - \vec{b} = 2(\hat{i} + \hat{j} + 2\hat{k}) - (3\hat{i} + 2\hat{j} - \hat{k}) = -\hat{i} + 0\hat{j} + 5\hat{k}$

$\therefore (\vec{a} + 3\vec{b}) \cdot (2\vec{a} - \vec{b}) = (10\hat{i} + 7\hat{j} - \hat{k}) \cdot (-\hat{i} + 0\hat{j} + 5\hat{k}) = 10(-1) + 7(0) + (-1)(5) = -10 - 5 = -15$

13. Find the value of λ if the vectors $\vec{a} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{b} = \hat{j} + \lambda\hat{k}$ are perpendicular.

The given vectors are perpendicular

$\therefore \quad \vec{a} \cdot \vec{b} = 0$

$(2\hat{i} - 3\hat{j} + \hat{k}) \cdot (\hat{j} + \lambda\hat{k}) = 0 \Rightarrow 0 - 3 + \lambda = 0 \Rightarrow \lambda = 3$

14. Write the cartesian equation of the plane whose vector equation is $\vec{r}.(2\hat{i} - \hat{j} + \hat{k}) = 7$.

We have, $\vec{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 7$

$\Rightarrow \quad (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} - \hat{j} + \hat{k}) = 7 \quad [\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$

$\Rightarrow \quad 2x - y + z = 7.$

15. Write the equation of line passing through the points (–1, 2, 1) and (3, 1, 4).

We know that equation of line passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

$\therefore$ Equation of line passing through $(-1, 2, 1)$ and $(3, 1, 4)$ is

$$\frac{x+1}{3+1} = \frac{y-2}{1-2} = \frac{z-1}{4-1} \Rightarrow \frac{x+1}{4} = \frac{y-2}{-1} = \frac{z-1}{3}$$

16. If $|\vec{a} \times \vec{b}|^2 + (\vec{a}.\vec{b})^2 = 144$ and $|\vec{a}| = 4$, then find the value of $|\vec{b}|$.

We know that, $|\vec{a} \times \vec{b}| + (\vec{a} \cdot \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2$

$\Rightarrow \quad 144 = (4)^2 |\vec{b}|^2$

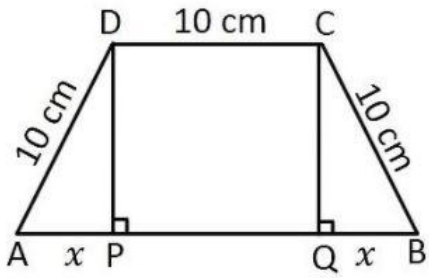
$\Rightarrow \quad |\vec{b}|^2 = \frac{144}{16} = 9$

$\therefore \quad |\vec{b}| = 3$

SECTION-II

Case study-based questions are compulsory. Attempt any four sub parts of each question. Each subpart carries 1 mark

17. A building has front gate has the figure as shown below:
It is in the shape of trapezium whose three sides other than base in 10 m. Height of the gate is 'h' m.



On the basis of above figure answer the following questions:

(i) The relation between a and h is

(a) $x^2 + h^2 = 10$ (b) $x^2 + h^2 = 100$ (c) $h^2 - x^2 = 10$ (d) $x^2 - h^2 = 10$

Ans: (b) $x^2 + h^2 = 100$

(ii) The are of gate A expressed as a function of x is

(a) $(10+x)\sqrt{100+x^2}$ (b) $(10-x)\sqrt{100+x^2}$

(c) $(10+x)\sqrt{100-x^2}$ (d) $(10-x)\sqrt{100-x^2}$

Ans: (c) $(10+x)\sqrt{100-x^2}$

(iii) The value of x when A is maximum is

(a) 5 m (b) 10 m (c) 15 m (d) 20 m

Ans: (a) 5 m

(iv) The value of h when A is maximum is

(a) $5\sqrt{2}$ m (b) $5\sqrt{3}$ m (c) $10\sqrt{2}$ m (d) $10\sqrt{3}$ m

Ans: (b) $5\sqrt{3}$ m

(v) Maximum value of A is (in m^2) is

(a) $\frac{75\sqrt{3}}{2}$ (b) $75\sqrt{3}$ (c) $\frac{75\sqrt{3}}{4}$ (d) 75

Ans: (a) $\frac{75\sqrt{3}}{2}$

18. The probability distribution functions which shows the number of hours (X) a student study during lockdown period in a day, is given by (where C > 0)

X	0	1	2
P(X)	$3C^3$	$4C - 10C^2$	$5C - 1$

On the basis of above information answer the following questions:

(i) The correct equation for C is

(a) $3C^3 - 10C^2 + C - 2 = 0$ (b) $3C^3 + 10C^2 + C - 2 = 0$

(c) $3C^3 - 10C^2 + 9C - 2 = 0$ (d) None of these

Ans: (c) $3C^3 - 10C^2 + 9C - 2 = 0$

(ii) The value of C is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{6}$

Ans: (b) $\frac{1}{3}$

(iii) $P(X < 2) =$

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{6}$

Ans: (b) $\frac{1}{3}$

(iv) $P(X = 1) =$

(a) $\frac{2}{9}$ (b) $\frac{1}{9}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

Ans: (a) $\frac{2}{9}$

(v) $P(X \geq 0) =$

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) 1 (d) 0

Ans: (c) 1

PART B

SECTION – III

Questions 19 to 28 carry 2 marks each.

19. An urn contains 5 white and 8 white black balls. Two successive drawing of three balls at a time are made such that the balls are not replaced before the second draw. Find the probability that the first draw gives 3 white balls and second draw gives 3 black balls.

Consider the following events

A = Drawing 3 white balls in first draw

B = Drawing 3 black balls in the second draw.

Required probability $= P(A \cap B) = P(A)\,P(B/A)$ ...(i)

Now, $P(A) = \frac{^5C_3}{^{13}C_3} = \frac{10}{286} = \frac{5}{143}$

After drawing 3 white balls in first draw 10 balls are left in the bag, out of which 8 are black balls.

$\therefore P(B/A) = \frac{^8C_3}{^{10}C_3} = \frac{56}{120} = \frac{7}{15}$

Substituting these values in Eq. (i), we get

Required probability $= P(A \cap B) = P(A)\,P(B/A)$

$$= \frac{5}{143} \times \frac{7}{15} = \frac{7}{429}$$

20. Find X and Y, if X + Y = $\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and X – Y = $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$.

We have, $X+Y=\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}$ and $X-Y=\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

$$\therefore \quad (X+Y)+(X-Y)=\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}+\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow 2X=\begin{bmatrix} 7+3 & 0+0 \\ 2+0 & 5+3 \end{bmatrix}=\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix} \Rightarrow \quad X=\frac{1}{2}\begin{bmatrix} 10 & 0 \\ 2 & 8 \end{bmatrix}=\begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$$

and $\quad (X+Y)-(X-Y)=\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}-\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}=\begin{bmatrix} 7 & 0 \\ 2 & 5 \end{bmatrix}+\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$

$$\Rightarrow 2Y=\begin{bmatrix} 7-3 & 0+0 \\ 2+0 & 5-3 \end{bmatrix}=\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix} \Rightarrow \quad Y=\frac{1}{2}\begin{bmatrix} 4 & 0 \\ 2 & 2 \end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

Hence, $X=\begin{bmatrix} 5 & 0 \\ 1 & 4 \end{bmatrix}$ and $Y=\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

21. Examine the continuity of the function: $f(x)=\begin{cases} \dfrac{|x-4|}{2(x-4)}, & x\neq 4 \\ 0 & , x=4 \end{cases}$ at x = 4.

At $x=4$, $\text{LHL}=\lim_{h\to 0} f(4-h)=\lim_{h\to 0}\frac{|4-h-4|}{2(4-h-4)}$

$$=\lim_{h\to 0}\frac{|-h|}{2(-h)}=\lim_{h\to 0}\frac{h}{-2h}=\lim_{h\to 0}\left(-\frac{1}{2}\right)=-\frac{1}{2}$$

$\text{RHL}=\lim_{h\to 0} f(4+h)=\lim_{h\to 0}\frac{|4+h-4|}{2(4+h-4)}$

$$=\lim_{h\to 0}\frac{|h|}{2h}=\lim_{h\to 0}\frac{h}{2h}=\lim_{h\to 0}\left(\frac{1}{2}\right)=\frac{1}{2}$$

and $f(4)=0$

$\text{LHL}\neq\text{RHL}\neq f(4)$

Hence, the given function is discontinuous at $x=4$.

22. If $y=\dfrac{1}{\sqrt{a^2-x^2}}$, then find $\dfrac{dy}{dx}$.

Given, $y=\dfrac{1}{\sqrt{a^2-x^2}}$

On putting $u=a^2-x^2$, we get $y=\dfrac{1}{\sqrt{u}}=u^{-1/2}$ and $u=a^2-x^2$

$$\therefore \quad \frac{dy}{du}=-\frac{1}{2}u^{-3/2} \text{ and } \frac{du}{dx}=-2x$$

Now, $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}=-\dfrac{1}{2}u^{-3/2}\times(-2x)=-\dfrac{1}{2u^{3/2}}\times(-2x)=\dfrac{x}{(a^2-x^2)^{3/2}}$

23. Find the value of $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)-2\sec^{-1}\left(2\tan\dfrac{\pi}{6}\right)$.

Let $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\theta_1 \Rightarrow \sin\theta_1 = -\frac{\sqrt{3}}{2} \Rightarrow \sin\theta_1 = -\sin\frac{\pi}{3}$

$\Rightarrow \sin\theta_1 = \sin\left(-\frac{\pi}{3}\right) \Rightarrow \theta_1 = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

and $\sec^{-1}\left(2\tan\frac{\pi}{6}\right)=\theta_2 \Rightarrow \sec^{-1}\left(\frac{2}{\sqrt{3}}\right)=\theta_2 \Rightarrow \sec\theta_2 = \frac{2}{\sqrt{3}}$

$\Rightarrow \sec\theta_2 = \sec\left(\frac{\pi}{6}\right) \Rightarrow \theta_2 = \frac{\pi}{6}$

$\Rightarrow \theta_2 = \sec^{-1}\left(2\tan\frac{\pi}{6}\right) = \frac{\pi}{6}$

$\therefore \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) - 2\sec^{-1}\left(2\tan\frac{\pi}{6}\right) = -\frac{\pi}{3} - 2\times\frac{\pi}{6} = -\frac{\pi}{3} - \frac{\pi}{3} = -\frac{2\pi}{3}$

24. Given $|\vec{a}|=10$, $|\vec{b}|=2$ and $\vec{a}.\vec{b}=12$, then find $|\vec{a}\times\vec{b}|$.

We know that, $(\vec{a}\,.\,\vec{b})^2 + |\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2$

$\therefore 12^2 + |\vec{a}\times\vec{b}|^2 = (10)^2\times(2)^2 \quad [\because \vec{a}\cdot\vec{b}=12, |\vec{a}|=10, |\vec{b}|=2]$

$\Rightarrow 144 + |\vec{a}\times\vec{b}|^2 = 400 \Rightarrow |\vec{a}\times\vec{b}|^2 = 400-144 \Rightarrow |\vec{a}\times\vec{b}|^2 = 256 \Rightarrow |\vec{a}\times\vec{b}| = 16$

25. If x = 10(t – sin$t$), y = 12(1 – cos$t$), find $\frac{dy}{dx}$.

We have, $x = 10(t - \sin t)$ and $y = 12(1-\cos t)$

$\therefore \frac{dx}{dt} = 10(1-\cos t)$ and $\frac{dy}{dt} = 12(\sin t)$

$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12\sin t}{10(1-\cos t)} = \frac{12\times 2\sin(t/2)\cos(t/2)}{10\times 2\sin^2(t/2)} = \frac{6}{5}\cot(t/2)$

26. Find the equation of the tangent to the curve y = –5x^2 + 6x + 7 at (1/2, 35/4).

The equation of the given curve is $y = -5x^2 + 6x + 7$

$\Rightarrow \frac{dy}{dx} = -10x + 6 \Rightarrow \left(\frac{dy}{dx}\right)_{(1/2,\ 35/4)} = -\frac{10}{2} + 6 = 1$

The required equation of the tangent at (1/2, 35/4) is

$$y - \frac{35}{4} = \left(\frac{dy}{dx}\right)\left(x-\frac{1}{2}\right) \Rightarrow y - \frac{35}{4} = 1\left(x - \frac{1}{2}\right) \Rightarrow 4(x-y)+33=0$$

27. Evaluate: $\int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$

We have, $I = \int_0^1 \frac{\tan^{-1}x}{1+x^2}dx$

Put $\tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}dx = dt$

Again, when $x=0 \Rightarrow t = \tan^{-1}0 = 0$

and when $x = 1 \Rightarrow t = \tan^{-1}1 = \frac{\pi}{4}$

$\therefore I = \int_0^{\frac{\pi}{4}} t\,dt = \left[\frac{t^2}{2}\right]_0^{\frac{\pi}{4}} = \frac{1}{2}\left[\left(\frac{\pi}{4}\right)^2 - 0\right] = \frac{\pi^2}{32}$

28. Two vectors $\vec{a}$ and $\vec{b}$, prove that the vector $|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ is orthogonal to the vector $|\vec{a}|\vec{b}-|\vec{b}|\vec{a}$.

Let $\vec{\alpha}=|\vec{a}|\vec{b}+|\vec{b}|\vec{a}$ and $\vec{\beta}=|\vec{a}||\vec{b}-|\vec{b}|\vec{a}$. Then,

$$\vec{\alpha}\cdot\vec{\beta}=\{|\vec{a}|\vec{b}+|\vec{b}|\vec{a}\}\cdot\{|\vec{a}|\vec{b}-|\vec{b}|\vec{a}\}$$

$\Rightarrow\ \vec{\alpha}\cdot\vec{\beta}=|\vec{a}|^2(\vec{b}\cdot\vec{b})-|\vec{a}||\vec{b}|(\vec{b}\cdot\vec{a})+|\vec{b}||\vec{a}|(\vec{a}\cdot\vec{b})-|\vec{b}|^2(\vec{a}\cdot\vec{a})$

$\Rightarrow\ \vec{\alpha}\cdot\vec{\beta}=|\vec{a}|^2|\vec{b}|^2-|\vec{a}||\vec{b}|(\vec{a}\cdot\vec{b})+|\vec{a}||\vec{b}|(\vec{a}\cdot\vec{b})-|\vec{b}|^2|\vec{a}|^2$

$\Rightarrow\ \vec{\alpha}\cdot\vec{\beta}=0$

$\therefore\ \vec{\alpha}$ is perpendicular (or orthogonal) to $\vec{\beta}$.

SECTION – IV

Questions 29 to 35 carry 3 marks each.

29. Let the function $f : R^+ \to [4,\infty)$ given by f(x) = x^2 + 4. Prove that f is bijective.

We have a mapping $f : R^+ \to [4,\infty)$ given by $f(x)=x^2+4$.

To prove f is invertible.

For *f* to be one-one

Let $x_1, x_2 \in R^+$ be any arbitrary elements, such that

$f(x_1)=f(x_2)\ \Rightarrow\ x_1^2+4=x_2^2+4\ \Rightarrow\ x_1^2-x_2^2=0$

$\Rightarrow\ (x_1-x_2)(x_1+x_2)=0$

$\Rightarrow\ x_1-x_2=0\quad [\because x_1+x_2\neq 0 \text{ as } x_1, x_2\in R^+]$

$\Rightarrow\ x_1=x_2$ So, f is one-one.

For *f* to be onto

Let $y\in[4,\infty)$ be any arbitrary element and let $y=f(x)$

Then, $y=x^2+4$

$x^2=y-4$

$x=\pm\sqrt{y-4}$

$\because x\in R^+$, therefore $x\neq-\sqrt{y-4}$.

Now, $x=\sqrt{y-4}\in R^+$

$[\because 4\le y<\infty \Rightarrow 0\le y-4<\infty \Rightarrow 0\le\sqrt{y-4}<\infty]$

Thus, for each $y\in[4,\infty)$, there exist $x=\sqrt{y-4}\in R^+$ such that $f(x)=y$

So, f is onto.

Hence, f is bijective.

30. Find the particular solution of differential equation $(3xy + y^2)dx + (x^2 + xy)dy = 0$ for x = 1 and y = 1.

We have, $(3xy + y^2)dx + (x^2 + xy)dy = 0$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{3xy + y^2}{x^2 + xy}\right) \quad \ldots(i)$$

Since, each of the functions $(3xy + y^2)$ and $(x^2 + xy)$ is a homogeneous function of degree 2, the given equation is, therefore a homogeneous differential equation.

On putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in Eq. (i) we get

$$v + x\frac{dv}{dx} = -\left\{\frac{3v\,x^2 + x^2v^2}{x^2 + vx^2}\right\} \Rightarrow v + x\frac{dv}{dx} = -\left\{\frac{3v + v^2}{1+v}\right\} \Rightarrow x\frac{dv}{dx} = -\left\{\frac{3v + v^2}{1+v} + v\right\}$$

$$\Rightarrow x\frac{dv}{dx} = -\left\{\frac{2v^2 + 4v}{v+1}\right\} \Rightarrow (v+1)x\,dv = -(2v^2 + 4v)dx \Rightarrow \frac{v+1}{2v^2 + 4v}dv = -\frac{dx}{x}$$

[by separating variables]

$$\Rightarrow \frac{(2v+2)dv}{v^2 + 2v} = -4\frac{dx}{x} \Rightarrow \frac{1+v^3}{v^4}dv = -\frac{dx}{x} \text{ [by separating variables]}$$

$$\Rightarrow \left(\frac{1}{v^4} + \frac{1}{v}\right)dv = -\frac{dx}{x} \Rightarrow \frac{v^{-3}}{-3} + \log|v| = -\log|x| + C$$

[integrating both sides]

$$\Rightarrow -\frac{1}{3v^3} + \log|v| + \log|x| = C \Rightarrow -\frac{1}{3}\frac{x^3}{y^3} + \log\left|\frac{y}{x}.x\right| = C \Rightarrow -\frac{x^3}{3y^3} + \log|y| = C$$

which is the required solution.

31. Evaluate: $\int_0^{\pi/2} \log(\sin x)dx$

Let $I = \int_0^{\pi/2} \log(\sin x)dx \quad \ldots(i)$

$\therefore \quad I = \int_0^{\pi/2} \log\left(\sin\left(\frac{\pi}{2} - x\right)\right)dx = \int_0^{\pi/2} \log(\cos x)dx \quad \ldots(ii)$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \{\log(\sin x) + \log(\cos x)\}\,dx = \int_0^{\pi/2} (\log(\sin x \cdot \cos x))dx$$

$$= \int_0^{\pi/2} \log\left(\frac{2\sin x\cos x}{2}\right)dx \Rightarrow 2I = \int_0^{\pi/2} \log(\sin 2x)dx - \int_0^{\pi/2} \log 2\,dx \quad \ldots(iii)$$

On putting $\quad 2x = t \Rightarrow 2dx = dt \Rightarrow dx = \frac{1}{2}dt$

When $\quad x = 0$, then $t = 0$ and $x = \frac{\pi}{2}$, then $t = \pi$

From Eq. (iii), we get

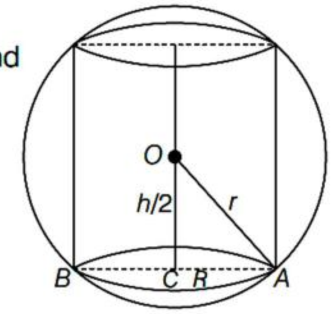
Now, $\int_0^{\pi/2} \log(\sin 2x)dx = \int_0^{\pi} \frac{1}{2}(\log(\sin t))dt = \frac{2}{2}\int_0^{\pi/2} [\log(\sin t)]dt = \int_0^{\pi/2} \log(\sin t)dt = I$

From Eq. (iii), we get $2I = I - \int_0^{\pi/2} \log 2\,dx \Rightarrow I = [-\log(2)\,x]_0^{\pi/2} = -\frac{\pi}{2}\log 2$

32. Find the volume of the largest cylinder that can be inscribed in sphere of radius 'r'.

Given, r is the radius of sphere.

Let R be the radius, h be the height of cylinder and V be the volume of the cylinder.



Then, $V = \pi R^2 h$ ...(i)

In right angled ΔOAC, we have $r^2 = R^2 + \left(\frac{h}{2}\right)^2$

$\Rightarrow R^2 = r^2 - \frac{h^2}{4}$

$\therefore V = \pi r^2 h - \frac{\pi h^3}{4}$

$\therefore \frac{dV}{dh} = \pi r^2 - \frac{3\pi h^2}{4}$

Now, $\frac{d^2V}{dh^2} = 0 - \frac{6\pi h}{4} = \frac{-3\pi h}{2}$

For maximum or minimum value of V,

put $\frac{dV}{dh} = 0 \Rightarrow \pi r^2 - \frac{3\pi h^2}{4} = 0 \Rightarrow h = \frac{2}{\sqrt{3}} r$

Now, $\left(\frac{d^2V}{dh^2}\right)_{h=\frac{2}{\sqrt{3}}r} = \frac{-3\pi}{2} \times \frac{2}{\sqrt{3}} r = -\sqrt{3}\pi r < 0$

Thus, V is maximum when $h = \frac{2}{\sqrt{3}} r$.

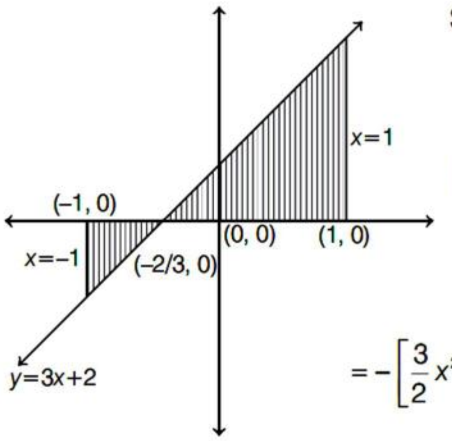
$\therefore R$ is calculated as

$$R^2 = r^2 - \frac{h^2}{4} \Rightarrow R^2 = r^2 - \frac{1}{4} \times \left(\frac{2}{\sqrt{3}} r\right)^2 = \sqrt{\frac{2}{3}} r$$

$\therefore$ Maximum volume of the cylinder is given by

$$V_{max} = \pi R^2 h = \pi \left(\sqrt{\frac{2}{3}} r\right)^2 \left(\frac{2}{\sqrt{3}} r\right) = \frac{4\pi r^3}{3\sqrt{3}} \text{ cu units}$$

33. Find the area of the region bounded by the line y = 3x + 2, the x-axis and the ordinates x = –1 and x = 1.



So, required area is given by

$$= \int_{-1}^{1} |y|\, dx = \int_{-1}^{-2/3} |y|\, dx + \int_{-2/3}^{1} |y|\, dx = \int_{-1}^{-2/3} -y\, dx + \int_{-2/3}^{1} y\, dx$$

$[\because y < 0 \text{ for } -1 < x < -\frac{2}{3} \text{ and } y > 0 \text{ for } -\frac{2}{3} < x < 1]$

$$= -\int_{-1}^{-2/3} (3x+2)\, dx + \int_{-2/3}^{1} (3x+2)\, dx$$

$$= -\left[\frac{3}{2}x^2 + 2x\right]_{-1}^{-2/3} + \left[\frac{3x^2}{2} + 2x\right]_{-2/3}^{1} = \frac{1}{6} + \frac{25}{6} = \frac{13}{3} \text{ sq. units.}$$

34. Evaluate: $\int \frac{x^2}{x \sin x + \cos x} dx$

Let $I = \int \frac{x^2}{(x\sin x + \cos x)^2}dx \Rightarrow I = \int \frac{x\cos x}{(x\sin x + \cos x)^2}\cdot x\sec x\,dx$ ...(i)

Put $x\sin x + \cos x = t \Rightarrow (x\cos x + \sin x - \sin x)dx = dt \Rightarrow x\cos x\,dx = dt$

$\therefore I_1 = \int \frac{x\cos x}{(x\sin x + \cos x)^2}dx = \int \frac{dt}{t^2} = \frac{-1}{t} = \frac{-1}{x\sin x + \cos x}$ [put $t = x\sin x + \cos x$]

Now, integrating Eq. (i) by parts, we get

$$I = \int \underset{\text{I}}{x\sec x}\cdot \underset{\text{II}}{\frac{x\cos x}{(x\sin x + \cos x)^2}}dx = x\sec x\cdot\frac{(-1)}{x\sin x + \cos x} - \int (1\cdot\sec x + x\sec x\tan x)\cdot\frac{-dx}{x\sin x + \cos x}$$

$$= \frac{-x\sec x}{x\sin x + \cos x} + \int \sec x\left(1 + \frac{x\sin x}{\cos x}\right)\frac{dx}{x\sin x + \cos x}$$

$$= \frac{-x\sec x}{x\sin x + \cos x} + \int \sec^2 x\,dx = \frac{-x\sec x}{x\sin x + \cos x} + \tan x + C$$

35. Find the point on the curve $y^2 = 2x$ which is at a minimum distance from the point (1, 4).

The given equation of curve is $y^2 = 2x$ and the given point is Q (1, 4).

Let $P(x, y)$ be any point on the curve.

Now, distance between points P and Q is given by

$PQ = \sqrt{(1-x)^2 + (4-y)^2}$

$\Rightarrow PQ = \sqrt{1 + x^2 - 2x + 16 + y^2 - 8y}$

$= \sqrt{x^2 + y^2 - 2x - 8y + 17}$

On squaring both sides, we get

$PQ^2 = x^2 + y^2 - 2x - 8y + 17$

$\Rightarrow PQ^2 = \left(\frac{y^2}{2}\right)^2 + y^2 - 2\left(\frac{y^2}{2}\right) - 8y + 17$

$\left[\text{given, } y^2 = 2x \Rightarrow x = \frac{y^2}{2}\right]$

$\therefore PQ^2 = \frac{y^4}{4} + y^2 - y^2 - 8y + 17$

$\Rightarrow PQ^2 = \frac{y^4}{4} - 8y + 17$

Let $PQ^2 = Z$ Then, $Z = \frac{y^4}{4} - 8y + 17$

On differentiating both sides w.r.t. y, we get

$\frac{dZ}{dy} = \frac{4y^3}{4} - 8 = y^3 - 8$

For maxima or minima, put $\frac{dZ}{dy} = 0$

$\Rightarrow y^3 - 8 = 0 \Rightarrow y^3 = 8 \Rightarrow y = 2$

Also, $\frac{d^2Z}{dy^2} = \frac{d}{dy}(y^3 - 8) = 3y^2$

On putting $y = 2$, we get $\left(\frac{d^2Z}{dy^2}\right)_{y=2} = 3(2)^2 = 12 > 0$

$\because \frac{d^2Z}{dy^2} > 0$

$\therefore Z$ is minimum and therefore PQ is also minimum as $Z = PQ^2$.

On putting $y = 2$ in the given equation, i.e. $y^2 = 2x$, we get

$(2)^2 = 2x$

$\Rightarrow 4 = 2x \Rightarrow x = 2$

Hence, the point which is at a minimum distance from point (1, 4) is P (2, 2).

SECTION – V

Questions 36 to 38 carry 5 marks each.

36. Solve the LPP graphically maximize, $Z = 1500(7x + 6y)$ subject to constraints;
$x + y \leq 50$; $2x + y \leq 80$, $x, y \geq 0$.

We have following LPP

Maximise $Z = 1500(7x + 6y)$

Subject to the constraints

$$x + y \le 50 \quad \text{...(i)}$$

$$2x + y \le 80 \quad \text{...(ii)}$$

$$\text{and} \quad x, y \ge 0 \quad \text{...(iii)}$$

Firstly, draw the graph of the line $x + y = 50$

x	0	50
y	50	0

Put (0, 0) in the inequality $x + y \le 50$, we get

$0 + 0 \le 50$, which is true.

So, the half plane is towards the origin.

Secondly, draw the graph of the line $2x + y = 80$

x	0	40
y	80	0

Put (0, 0) in the inequality $2x + y \le 80$, we get

$2 \times 0 + 0 \le 80$

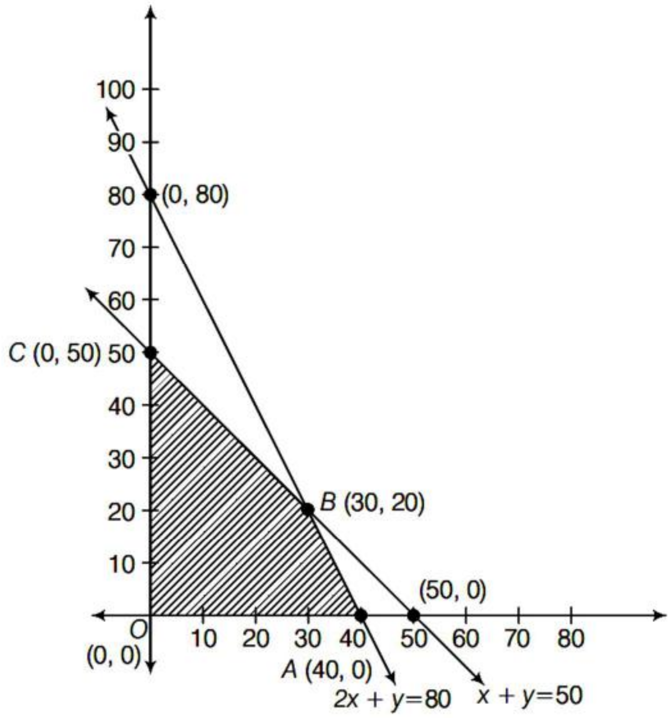
$\Rightarrow$ $0 \le 80$, which is true

So, the half plane is towards the origin.

Since, $x, y \ge 0$ the feasible region lies in the first quadrant

On solving (i) and (ii), we get $B(30, 20)$.

The corner points of feasible region are $O(0, 0)$, $A(40, 0)$ $B(30, 20)$ and $C(0, 50)$.



The values of Z at these corner points are calculated as

Corner	$Z = 1500(7x + 6y)$
$O(0, 0)$	$Z = 0$
$A(40, 0)$	$Z = 1500(7 \times 40 + 6 \times 0) = 420000$
$B(30, 20)$	$Z = 1500(7 \times 30 + 6 \times 20)$ $= 495000$ (maximum)
$C(0, 50)$	$Z = 1500(7 \times 0 + 6 \times 50) = 450000$

The maximum value of Z is 495000 at (30,20).

37. Find the equation of plane determined by points A(3, –1, 2), B(5, 2, 4), C(–1. –1, 6) and hence find the distance between plane and point P(6, 5, 9).

Given points are $A(3, -1, 2)$, $B(5, 2, 4)$ and $C(-1, -1, 6)$.

Now, equation of plane passing through A, B and C is

given by $$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x - 3 & y + 1 & z - 2 \\ 5 - 3 & 2 + 1 & 4 - 2 \\ -1 - 3 & -1 + 1 & 6 - 2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x - 3 & y + 1 & z - 2 \\ 2 & 3 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 0$$

On expanding along R_1, we get $(x - 3)(12 - 0) - (y + 1)(8 + 8) + (z - 2)(0 + 12) = 0$

$\Rightarrow 12x - 36 - 16y - 16 + 12z - 24 = 0$

$\Rightarrow 12x - 16y + 12z = 76 \Rightarrow 3x - 4y + 3z = 19$

Now, distance of the point (6, 5, 9) from the plane (i) is

$$d = \left| \frac{3(6) - 4(5) + 3(9) - 19}{\sqrt{3^2 + 4^2 + 3^2}} \right| = \left| \frac{18 - 20 + 27 - 19}{\sqrt{9 + 16 + 9}} \right| = \left| \frac{6}{\sqrt{34}} \right| = \frac{6}{\sqrt{34}} \text{ units}$$

38. Evaluate the product AB, where $A=\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & 4\\ 0 & 1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}2 & 2 & -4\\ -4 & 2 & -4\\ 2 & -1 & 5\end{bmatrix}$. Hence solve the system of linear equations x – y = 3, 2x + 3y + 4z = 17 and y + 2y = 7.

$$AB=\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & 4\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}2 & 2 & -4\\ -4 & 2 & -4\\ 2 & -1 & 5\end{bmatrix}=\begin{bmatrix}2+4+0 & 2-2+0 & -4+4+0\\ 4-12+8 & 4+6-4 & -8-12+20\\ 0-4+4 & 0+2-2 & 0-4+10\end{bmatrix}=\begin{bmatrix}6 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 6\end{bmatrix}=6I$$

$\therefore\quad AB = 6I$

Given equations are $x - y + 0z = 3$

$2x + 3y + 4z = 17$

$0x + y + 2z = 7$

These can be written as $\begin{bmatrix}1 & -1 & 0\\ 2 & 3 & 4\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ 17\\ 7\end{bmatrix} \Rightarrow AX = C$ where $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, C=\begin{bmatrix}3\\ 17\\ 7\end{bmatrix}$

$\Rightarrow\quad X = A^{-1}C$

From (i), $AB = 6I \Rightarrow B = 6A^{-1}I \Rightarrow A^{-1} = \frac{1}{6}B$

$$\Rightarrow\quad A^{-1}=\frac{1}{6}\begin{bmatrix}2 & 2 & -4\\ -4 & 2 & -4\\ 2 & -1 & 5\end{bmatrix}$$

$$\therefore \text{ From (ii) } X=\frac{1}{6}\begin{bmatrix}2 & 2 & -4\\ -4 & 2 & -4\\ 2 & -1 & 5\end{bmatrix}\begin{bmatrix}3\\ 17\\ 7\end{bmatrix}$$

$$\Rightarrow\quad \begin{bmatrix}x\\ y\\ z\end{bmatrix}=\frac{1}{6}\begin{bmatrix}12\\ -6\\ 24\end{bmatrix}=\begin{bmatrix}2\\ -1\\ 4\end{bmatrix}$$

$\therefore\qquad x = 2, y = -1, z = 4.$